

Structure of Hyperbolic Unitary Groups II: Classification of E-normal Subgroups

Raimund Preusser

E-mail: preusser@math.uni-bielefeld.de

Abstract. This paper proves the sandwich classification conjecture for subgroups of an even dimensional hyperbolic unitary group $U_{2n}(R, \Lambda)$ which are normalized by the elementary subgroup $EU_{2n}(R, \Lambda)$, under the condition that R is a quasi-finite ring with involution, i.e a direct limit of module finite rings with involution, and $n \geq 3$.

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1 Introduction

This paper is a successor of the paper [4] by A. Bak and N. Vavilov. The main result is the following: If (R, Λ) is a form ring such that R is quasi-finite and H is a subgroup of the hyperbolic unitary group $U_{2n}(R, \Lambda)$ where $n \geq 3$, then

$$\begin{aligned} &H \text{ is normalized by the elementary subgroup } EU_{2n}(R, \Lambda) \text{ of } U_{2n}(R, \Lambda) \\ \Leftrightarrow &\exists! \text{ form ideal } (I, \Gamma) : EU_{2n}((R, \Lambda), (I, \Gamma)) \subseteq H \subseteq CU_{2n}((R, \Lambda), (I, \Gamma)) \end{aligned} \quad (1.1)$$

where $EU_{2n}((R, \Lambda), (I, \Gamma))$ denotes the relative elementary subgroup of level (I, Γ) and $CU_{2n}((R, \Lambda), (I, \Gamma))$ denotes the full congruence subgroup of level (I, Γ) . This result extends the range of validity of previous results. If R has finite Bass-Serre dimension d (cf.[1]) then the result was proved already in [2] provided $n \geq \sup(d+2, 3)$ and if R is commutative, it was proved recently in [9]. An incorrect proof, which can be repaired when 2 is invertible in R , was given in [7].

The paper is organized as follows. In section 2 we recall some standard notation which will be used throughout the paper. In section 3 we recall the definitions of the hyperbolic unitary group and some important subgroups. In section 4 we prove the main result (1.1), first for certain almost commutative rings and then for quasi-finite rings.

The current paper formed a part of my doctoral dissertation. I would like to thank my advisor Anthony Bak for his guidance during the preparation of my

dissertation and in particular for making me aware of the theory of model unitary groups, which is used in section 4.

2 Notation

Let G be a group and H, K be subsets of G . The subgroup of G generated by H is denoted by $\langle H \rangle$. If $g, h \in G$, let ${}^h g := hgh^{-1}$, $g^h := h^{-1}gh$ and $[g, h] := ghg^{-1}h^{-1}$. Set ${}^K H := \langle \{ {}^k h | h \in H, k \in K \} \rangle$ and $H^K := \langle \{ h^k | h \in H, k \in K \} \rangle$. Analogously define $[H, K]$ and HK . Instead of ${}^K \{g\}$ we write ${}^K g$ (analogously we write g^K instead of $\{g\}^K$, ${}^g H$ instead of $\{g\}H$, $[g, K]$ instead of $[\{g\}, K]$ etc.).

In this paper, ring will always mean associative ring with 1 such that $1 \neq 0$. Ideal will mean two-sided ideal. If R is a ring and $m, n \in \mathbb{N}$, then the set of all invertible elements in R is denoted by R^* and the set of all $m \times n$ matrices with entries in R is denoted by $M_{m \times n}(R)$. If $a \in M_{m \times n}(R)$, let $a_{ij} \in R$ denote the element in the (i, j) 'th position. Let $a^t \in M_{n \times m}(R)$ denote its transpose, thus $(a^t)_{ij} = a_{ji}$. Denote the i -th row of a by a_{i*} and the j -th column of a by a_{*j} . We set $M_n(R) := M_{n \times n}(R)$. The identity matrix in $M_n(R)$ is denoted by e or $e^{n \times n}$ and the matrix with a 1 at position (i, j) and zeros elsewhere is denoted by e^{ij} . If $a \in M_n(R)$ is invertible, the entry of a^{-1} at position (i, j) is denoted by a'_{ij} , the i -th row of a^{-1} by a'_{i*} and the j -th column of a^{-1} by a'_{*j} . Further we denote by ${}^n R$ the set of all rows $v = (v_1, \dots, v_n)$ with entries in R and by R^n the set of all columns $u = (u_1, \dots, u_n)^t$ with entries in R .

3 Bak's hyperbolic unitary groups

In order to classify the subgroups of a general linear group normalized by its elementary subgroup, the notion of an ideal in a ring is sufficient. Bak's dissertation [2] showed that the notion of an ideal by itself was not sufficient to solve the analogous classification problem for unitary groups, but that a refinement of the notion an ideal, called a form ideal, was necessary. This led naturally to a more general notion of unitary group, which was defined over a form ring instead of just a ring and generalized all previous concepts. We describe form rings (R, Λ) and form ideals (I, Γ) first, then hyperbolic unitary groups $U_{2n}(R, \Lambda)$ over form rings (R, Λ) . For form ideals (I, Γ) , we recall the definitions of the following subgroups of $U_{2n}(R, \Lambda)$; the preelementary groups $EU_{2n}(I, \Gamma)$, the relative elementary groups $EU_{2n}((R, \Lambda), (I, \Gamma))$, the principal congruence subgroups $U_{2n}((R, \Lambda), (I, \Gamma))$, and the full congruence subgroups $CU_{2n}((R, \Lambda), (I, \Gamma))$.

Definition 3.1 Let R be a ring and

$$\begin{aligned} \bar{\cdot} : R &\rightarrow R \\ r &\mapsto \bar{r} \end{aligned}$$

an involution on R , i.e. $\overline{\bar{r} + \bar{s}} = \bar{r} + \bar{s}$, $\overline{\bar{r}s} = \bar{s}\bar{r}$ and $\bar{\bar{r}} = r$ for any $r, s \in R$. Let

$\lambda \in \text{center}(R)$ such that $\lambda\bar{\lambda} = 1$ and set $\Lambda_{\min} := \{r - \lambda\bar{r} | r \in R\}$ and $\Lambda_{\max} := \{r \in R | r = -\lambda\bar{r}\}$. An additive subgroup Λ of R such that

$$(1) \quad \Lambda_{\min} \subseteq \Lambda \subseteq \Lambda_{\max} \text{ and}$$

$$(2) \quad r\Lambda\bar{r} \subseteq \Lambda \quad \forall r \in R$$

is called a *form parameter*. If Λ is a form parameter for R , the pair (R, Λ) is called a *form ring*.

Definition 3.2 Let (R, Λ) be a form ring and I an ideal such that $\bar{I} = I$. Set $\Gamma_{\max} = I \cap \Lambda$ and $\Gamma_{\min} = \{\xi - \lambda\bar{\xi} | \xi \in I\} + \langle \{\zeta\alpha\bar{\zeta} | \zeta \in I, \alpha \in \Lambda\} \rangle$. If we want to stress that Γ_{\max} (resp. Γ_{\min}) belongs to I , we write Γ_{\max}^I (resp. Γ_{\min}^I). An additive subgroup Γ of I such that

$$(1) \quad \Gamma_{\min} \subseteq \Gamma \subseteq \Gamma_{\max} \text{ and}$$

$$(2) \quad \alpha\Gamma\bar{\alpha} \subseteq \Gamma \quad \forall \alpha \in R$$

is called a *relative form parameter of level I* . If Γ is a relative form parameter of level I , then (I, Γ) is called a *form ideal* of (R, Λ) .

Until the end of this section let $n \in \mathbb{N}$, (R, Λ) a form ring and (I, Γ) a form ideal of (R, Λ) .

Definition 3.3 Let V be a free right R -module of rank $2n$ and $B = (e_1, \dots, e_n, e_{-n}, \dots, e_{-1})$ an ordered basis of V . Let $\phi_B : V \rightarrow R^{2n}$ be the module isomorphism mapping e_i to the column whose i -th coordinate is one and all the other coordinates are zero if $1 \leq i \leq n$ and the column whose $(2n + 1 + i)$ -th coordinate is one and all the other coordinates are zero if $-n \leq i \leq -1$. In the following we will identify elements $v \in V$ with their images $\phi_B(v) \in R^{2n}$. Let $p \in M_n(R)$ be the matrix with ones on the skew diagonal and zeros elsewhere. We define the maps

$$\mathbb{f} : V \times V \rightarrow R$$

$$(v, w) \mapsto \bar{v}^t \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} w,$$

$$\mathbb{h} : V \times V \rightarrow R$$

$$(v, w) \mapsto \bar{v}^t \begin{pmatrix} 0 & p \\ \lambda p & 0 \end{pmatrix} w$$

and

$$\mathbb{q} : V \rightarrow R/\Lambda$$

$$v \mapsto \mathbb{f}(v, v) + \Lambda$$

where \bar{v} is obtained from v by applying $\bar{}$ to each entry of v . The maps \mathbb{f} , \mathbb{h} and \mathbb{q} are denoted in [4], page 164, by f , h and q , respectively. It is easy to check that

$\mathbb{f}(v, w) = \overline{v}_1 w_{-1} + \dots + \overline{v}_n w_{-n}$, $\mathbb{h}(v, w) = \overline{v}_1 w_{-1} + \dots + \overline{v}_n w_{-n} + \lambda \overline{v}_{-n} w_n + \dots + \lambda \overline{v}_{-1} w_1 = \mathbb{f}(v, w) + \lambda \overline{\mathbb{f}(w, v)}$ and $\mathbb{q}(v) = \overline{v}_1 v_{-1} + \dots + \overline{v}_n v_{-n} + \Lambda$ for any $v, w \in V$. For any $v \in V$, $\mathbb{f}(v, v)$ is called the *length* of v and is denoted by $|v|$.

Definition 3.4 The subgroup $U_{2n}(R, \Lambda) := \{\sigma \in GL(V) | (\mathbb{h}(\sigma u, \sigma v) = \mathbb{h}(u, v)) \wedge (\mathbb{q}(\sigma u) = \mathbb{q}(u)) \ \forall u, v \in V\}$ of $GL(V)$ is called the *hyperbolic unitary group*. We will identify $U_{2n}(R, \Lambda)$ with its image in $GL_{2n}(R)$ under the isomorphism $GL(V) \rightarrow GL_{2n}(R)$ determined by the ordered basis $(e_1, \dots, e_n, e_{-n}, \dots, e_{-1})$.

Definition 3.5 Let $\sigma \in M_n(R)$. By definition σ^* is the matrix in $M_n(R)$ whose entry at position (i, j) equals $\overline{\sigma}_{ji}$. Further we define $AH_n(R, \Lambda) := \{a \in M_n(R) | a = -\lambda a^*, a_{ii} \in \Lambda \ \forall i \in \{1, \dots, n\}\}$.

Lemma 3.6 Let (R, Λ) be a form ring, $n \in \mathbb{N}$ and $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_{2n}(R)$, where $a, b, c, d \in M_n(R)$. Then $\sigma \in U_{2n}(R, \Lambda)$ if and only if

$$(1) \ \sigma^{-1} = \begin{pmatrix} pd^*p & \overline{\lambda}pb^*p \\ \lambda pc^*p & pa^*p \end{pmatrix} \text{ and}$$

$$(2) \ a^*pc, b^*pd \in AH_n(R, \Lambda).$$

Proof See [4], p.166.

Remark

- (1) If $a \in M_n(R)$, then pa^*p is the matrix one gets by applying the involution to each entry of a and mirroring all entries on the skew diagonal.
- (2) In [2], [8] and [9] the ordered basis $(e_1, \dots, e_n, e_{-1}, \dots, e_{-n})$ is used and hence the matrices may look different. Let $\sigma \in GL(V)$. If the image of σ under the isomorphism $GL(V) \rightarrow GL_{2n}(R)$ determined by the ordered basis $(e_1, \dots, e_n, e_{-1}, \dots, e_{-n})$ (which is used in the papers mentioned above) equals $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in M_n(R)$, then the image of σ under the isomorphism $GL(V) \rightarrow GL_{2n}(R)$ determined by the ordered basis $(e_1, \dots, e_n, e_{-n}, \dots, e_{-1})$ (which is used in this paper) equals $\begin{pmatrix} a & bp \\ pc & pdp \end{pmatrix}$.

Definition 3.7 We define $\Omega_+ := \{1, \dots, n\}$, $\Omega_- := \{-n, \dots, -1\}$, $\Omega := \Omega_+ \cup \Omega_-$ and

$$\epsilon : \Omega \rightarrow \{-1, 1\}$$

$$i \mapsto \epsilon(i) := \begin{cases} 1, & \text{if } i \in \Omega_+, \\ -1, & \text{if } i \in \Omega_-. \end{cases}$$

Lemma 3.8 Let $\sigma \in GL_{2n}(R)$. Then $\sigma \in U_{2n}(R, \Lambda)$ if and only if

$$(1) \ \sigma'_{ij} = \lambda^{(\epsilon(j) - \epsilon(i))/2} \overline{\sigma}_{-j, -i} \ \forall i, j \in \{1, \dots, -1\} \text{ and}$$

(2) $|\sigma_{*j}| \in \Lambda \ \forall j \in \{1, \dots, -1\}$. ($|\sigma_{*j}| = \sum_{i=1}^n \bar{\sigma}_{ij} \sigma_{-i,j}$ is defined just before 3.4.)

Proof See [4], p.167.

Lemma 3.9 Let $\sigma \in U_{2n}(R, \Lambda)$, $x \in R^*$ and $k \in \{1, \dots, -1\}$. Then the statements below are true where $f_l := e_l^t$ for any $l \in \{1, \dots, -1\}$.

- (1) If the k -th column of σ equals xe_k then the $(-k)$ -th row of σ equals $\overline{x^{-1}}f_{-k}$.
- (2) If the k -th row of σ equals xf_k then the $(-k)$ -th column of σ equals $\overline{x^{-1}}e_{-k}$.

Proof

- (1) Since $\sigma^{-1}\sigma = e$ it follows that

$$(\sigma^{-1}\sigma)_{ij} = \sum_{l=1}^{-1} \sigma'_{il} \sigma_{lj} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \quad (3.1)$$

This implies that $1 = \sum_{l=1}^{-1} \sigma'_{kl} \sigma_{lk} = \sigma'_{kk} \sigma_{kk} = \sigma'_{kk} x$. Thus $\sigma'_{kk} = x^{-1}$. By Lemma 3.8, it follows that $\sigma_{-k, -k} = \overline{x^{-1}}$. On the other hand (3.1) implies that $0 = \sum_{l=1}^{-1} \sigma'_{il} \sigma_{lk} = \sigma'_{ik} \sigma_{kk} = \sigma'_{ik} x \ \forall i \in \{1, \dots, -1\} \setminus \{k\}$. It follows that $\sigma'_{ik} = 0 \ \forall i \in \{1, \dots, -1\} \setminus \{k\}$ and hence, by Lemma 3.8, $\sigma_{-k, -i} = 0 \ \forall i \in \{1, \dots, -1\} \setminus \{k\}$, i.e. $\sigma_{-k, i} = 0 \ \forall i \in \{1, \dots, -1\} \setminus \{-k\}$.

- (2) Since $\sigma\sigma^{-1} = e$ it follows that

$$(\sigma\sigma^{-1})_{ij} = \sum_{l=1}^{-1} \sigma_{il} \sigma'_{lj} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \quad (3.2)$$

This implies $1 = \sum_{l=1}^{-1} \sigma_{kl} \sigma'_{lk} = \sigma_{kk} \sigma'_{kk} = x \sigma'_{kk}$. Thus $\sigma'_{kk} = x^{-1}$. By Lemma 3.8, it follows that $\sigma_{-k, -k} = \overline{x^{-1}}$. On the other hand (3.2) implies that $0 = \sum_{l=1}^{-1} \sigma_{kl} \sigma'_{lj} = \sigma_{kk} \sigma'_{kj} = x \sigma'_{kj} \ \forall j \in \{1, \dots, -1\} \setminus \{k\}$. It follows that $\sigma'_{kj} = 0 \ \forall j \in \{1, \dots, -1\} \setminus \{k\}$ and hence, by Lemma 3.8, $\sigma_{-j, -k} = 0 \ \forall j \in \{1, \dots, -1\} \setminus \{k\}$, i.e. $\sigma_{j, -k} = 0 \ \forall j \in \{1, \dots, -1\} \setminus \{-k\}$. \square

Definition 3.10 If $i, j \in \Omega$ such that $i \neq \pm j$ and $\xi \in R$, then the matrix

$$T_{ij}(\xi) := e + \xi e^{ij} - \lambda^{(\epsilon(j) - \epsilon(i))/2} \bar{\xi} e^{-j, -i} \in U_{2n}(R, \Lambda)$$

is called an *elementary short root element*. If $i \in \Omega$ and $\alpha \in \lambda^{-(\epsilon(i)+1)/2} \Lambda$, then the matrix

$$T_{i, -i}(\alpha) := e + \alpha e^{i, -i} \in U_{2n}(R, \Lambda)$$

is called an *elementary long root element*. If $\sigma \in U_{2n}(R, \Lambda)$ is an elementary short root element or an elementary long root element, it is called an *elementary unitary*

matrix. The subgroup of $U_{2n}(R, \Lambda)$ generated by all elementary unitary matrices is called the *elementary unitary group* and is denoted by $EU_{2n}(R, \Lambda)$. Let $T_{ij}(\xi)$ be an elementary unitary matrix. If $i \neq -j \wedge \xi \in I$ or $i = -j \wedge \xi \in \lambda^{-(\epsilon(i)+1)/2}\Gamma$, then $T_{ij}(\xi)$ is called *elementary of level (I, Γ)* or (I, Γ) -*elementary*. The subgroup of $U_{2n}(R, \Lambda)$ generated by all (I, Γ) -elementary matrices is called the *preelementary subgroup of level (I, Γ)* and is denoted by $EU_{2n}(I, \Gamma)$. Its normal closure in $EU_{2n}(R, \Lambda)$ is called the *elementary subgroup of level (I, Γ)* and is denoted by $EU_{2n}((R, \Lambda), (I, \Gamma))$.

Definition 3.11 Let $i, j \in \{1, \dots, -1\}$ such that $i \neq \pm j$. Define $P_{ij} := e + e^{ij} - e^{ji} + \lambda^{(\epsilon(i)-\epsilon(j))/2}e^{-i,-j} - \lambda^{(\epsilon(j)-\epsilon(i))/2}e^{-j,-i} - e^{ii} - e^{jj} - e^{-i,-i} - e^{-j,-j} = T_{ij}(1)T_{ji}(-1)T_{ij}(1) \in EU_{2n}(R, \Lambda)$.

Lemma 3.12 *The relations*

$$T_{ij}(\xi) = T_{-j,-i}(-\lambda^{(\epsilon(j)-\epsilon(i))/2}\bar{\xi}), \quad (R1)$$

$$T_{ij}(\xi)T_{ij}(\zeta) = T_{ij}(\xi + \zeta), \quad (R2)$$

$$[T_{ij}(\xi), T_{hk}(\zeta)] = e, \quad (R3)$$

$$[T_{ij}(\xi), T_{jh}(\zeta)] = T_{ih}(\xi\zeta), \quad (R4)$$

$$[T_{ij}(\xi), T_{j,-i}(\zeta)] = T_{i,-i}(\xi\zeta - \lambda^{-\epsilon(i)}\bar{\zeta}\bar{\xi}) \text{ and } \quad (R5)$$

$$[T_{i,-i}(\alpha), T_{-i,j}(\xi)] = T_{ij}(\alpha\xi)T_{-j,j}(-\lambda^{(\epsilon(j)-\epsilon(-i))/2}\bar{\xi}\alpha\xi) \quad (R6)$$

hold where $h \neq j, -i$ and $k \neq i, -j$ in (R3), $i, h \neq \pm j$ and $i \neq \pm h$ in (R4) and $i \neq \pm j$ in (R5) and (R6).

Proof Straightforward calculation.

Definition 3.13 The group consisting of all $\sigma \in U_{2n}(R, \Lambda)$ such that $\sigma \equiv e \pmod{I}$ and $\mathbb{f}(\sigma u, \sigma u) \in \mathbb{f}(u, u) + \Gamma \ \forall u \in V$ is called the *principal congruence subgroup of level (I, Γ)* and is denoted by $U_{2n}((R, \Lambda), (I, \Gamma))$. By a theorem of Bak [2], 4.1.4, cf. [4], 4.4, it is a normal subgroup of $U_{2n}(R, \Lambda)$.

Lemma 3.14 Let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_{2n}(R, \Lambda)$, where $a, b, c, d \in M_n(R)$. Then $\sigma \in U_{2n}((R, \Lambda), (I, \Gamma))$ if and only if

(1) $\sigma \equiv e \pmod{I}$ and

(2) $|\sigma_{*j}| \in \Gamma \ \forall j \in \{1, \dots, -1\}$. ($|\sigma_{*j}| = \sum_{i=1}^n \bar{\sigma}_{ij}\sigma_{-i,j}$ is defined just before 3.4.)

Proof See [4], p.174.

Definition 3.15 The preimage of the center of $U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$ under the canonical homomorphism $U_{2n}(R, \Lambda) \rightarrow U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$ is called the *full congruence subgroup of level (I, Γ)* and is denoted by $CU_{2n}((R, \Lambda), (I, \Gamma))$.

Remark Obviously $U_{2n}((R, \Lambda), (I, \Gamma)) \subseteq CU_{2n}((R, \Lambda), (I, \Gamma))$ and $CU_{2n}((R, \Lambda), (I, \Gamma))$ is a normal subgroup of $U_{2n}(R, \Lambda)$.

Lemma 3.16 *If $n \geq 3$ and R is almost commutative (i.e. module finite over its center), then the equalities*

$$\begin{aligned} & [CU_{2n}((R, \Lambda), (I, \Gamma)), EU_{2n}(R, \Lambda)] \\ &= [EU_{2n}((R, \Lambda), (I, \Gamma)), EU_{2n}(R, \Lambda)] \\ &= EU_{2n}((R, \Lambda), (I, \Gamma)) \end{aligned}$$

hold.

Proof See [4], Theorem 1.1 and Lemma 5.2. □

4 Sandwich classification for hyperbolic unitary groups

In this section we prove our main results.

We begin by fixing notation which will be used for most of the section, up through the proof of Theorem 4.11. This theorem proves our main result over a special kind of almost commutative form ring, which will be described immediately below. The general result over quasi-finite form rings will be deduced from this result in Theorem 4.14. After fixing notation below, we shall explain the idea of the proof of Theorem 4.11 and how the rest of the section is organized.

Until the end of the proof of Theorem 4.11 let $n \geq 3$, (R, Λ) a form ring and C the subring of R consisting of all finite sums of elements of the form $c\bar{c}$ and $-c\bar{c}$ where c ranges over some subring $C' \subseteq \text{center}(R)$. Further assume that R is a Noetherian C -module. For any form ideal (I, Γ) of (R, Λ) and maximal ideal m of C set $S_m := C \setminus m$, $R_m := S_m^{-1}R$, $\Lambda_m := S_m^{-1}\Lambda$, $I_m := S_m^{-1}I$ and $\Gamma_m := S_m^{-1}\Gamma$. Let

$$\phi_m : U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma)) \rightarrow U_{2n}(R_m, \Lambda_m)/U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$$

be the homomorphism induced by F_m where

$$F_m : U_{2n}(R, \Lambda) \rightarrow U_{2n}(R_m, \Lambda_m)$$

is the homomorphism induced by the localisation homomorphism

$$f_m : R \rightarrow R_m.$$

Let

$$\psi : U_{2n}(R, \Lambda) \rightarrow U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$$

and

$$\psi_m : U_{2n}(R_m, \Lambda_m) \rightarrow U_{2n}(R_m, \Lambda_m)/U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$$

be the canonical homomorphisms. Further set $\lambda_m := f_m(\lambda)$. Note that the diagram

$$\begin{array}{ccc} U_{2n}(R, \Lambda) & \xrightarrow{\psi} & U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma)) \\ \downarrow F_m & & \downarrow \phi_m \\ U_{2n}(R_m, \Lambda_m) & \xrightarrow{\psi_m} & U_{2n}(R_m, \Lambda_m)/U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m)) \end{array}$$

is commutative for any form ideal (I, Γ) of (R, Λ) and maximal ideal m of C .

A road map of the proof of Theorem 4.11 is as follows. It is easy to show that there is a largest form ideal (I, Γ) of (R, Λ) such that $EU_{2n}((R, \Lambda), (I, \Gamma)) \subseteq H$. To show that $H \subseteq CU_{2n}((R, \Lambda), (I, \Gamma))$ we proceed by contradiction. We show that if $H \not\subseteq CU_{2n}((R, \Lambda), (I, \Gamma))$, then there is a form ideal (I', Γ') which properly contains (I, Γ) such that $EU_{2n}((R, \Lambda), (I', \Gamma')) \subseteq H \cdot U_{2n}((R, \Lambda), (I, \Gamma))$. Then it follows routinely from the standard mixed commutator formulas of [4], that $EU_{2n}((R, \Lambda), (I', \Gamma')) \subseteq H$, which contradicts the maximality of (I, Γ) . This is the approach pioneered by W. Klingenberg and H. Bass for the general linear group GL_n over respectively semilocal rings and rings R satisfying a stability condition and can also be applied over almost commutative rings. Over the last rings, the crucial step in the road map is embedding $GL_n(R)/GL_n(R, I)$ in the canonical way into $GL_n(R/I)$ and then showing that if $H' = \text{image of } H \text{ in } GL_n(R/I)$ is noncentral in $GL_n(R/I)$, then H' contains a relative elementary group $E_n(R/I, I'/I)$ for some ideal I' of R , which properly contains I . This result is established by a localization argument over the maximal ideals of $\text{center}(R/I)$. In the case of unitary groups, we would like to apply the same approach, but unfortunately this doesn't work, because for an arbitrary form ideal (I, Γ) , there is no unitary group in the sense of the current paper into which we can canonically embed $U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$. But, there is an unpublished theory of "model unitary groups" of A. Bak, which has the property that the quotient of any model unitary group by one of its congruence subgroups is again a model unitary group and which has an effective localization procedure. Of course, any unitary group $U_{2n}(R, \Lambda)$ is a model unitary group, but not conversely. In this section, we shall adapt the methods of model unitary groups to the circumstances at hand and show directly that the quotient group $U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$ has a good localization theory. The essential contributions here are made in Lemmas 4.2 and 4.4. Lemma 4.8 is an application of this localization theory and provides the key result needed to show that if $\psi(H) \not\subseteq \text{center}(U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma)))$ then there is a form ideal (I', Γ') which properly contains (I, Γ) such that $EU_{2n}((R, \Lambda), (I', \Gamma')) \subseteq H \cdot U_{2n}((R, \Lambda), (I, \Gamma))$, i.e. $\psi(EU_{2n}((R, \Lambda), (I', \Gamma'))) \subseteq \psi(H)$. The proof of Theorem 4.11 can be then concluded as above by applying the standard mixed commutator formulas.

Lemma 4.1 *Let (I, Γ) be a form ideal of (R, Λ) and $g' \in U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$ be noncentral. Then there is a maximal ideal m of C such that $I \cap C \subseteq m$ and $\phi_m(g')$ is noncentral.*

Proof Since $g' \in U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$ is noncentral, there is an $h' \in U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$ such that $g'h' \neq h'g'$. Let $g, h \in U_{2n}(R, \Lambda)$ such that $g' = gU_{2n}((R, \Lambda), (I, \Gamma))$ and $h' = hU_{2n}((R, \Lambda), (I, \Gamma))$. Set $\sigma := [g^{-1}, h^{-1}]$. Clearly $g'h' \neq h'g'$ implies $\sigma \notin U_{2n}((R, \Lambda), (I, \Gamma))$. Hence either $\sigma_{ij} \notin I$ for some $i, j \in \{1, \dots, -1\}$ such that $i \neq j$, or $\sigma_{ii} - 1 \notin I$ for some $i \in \{1, \dots, -1\}$ or $x_j := |\sigma_{*j}| \notin \Gamma$ for some $j \in \{1, \dots, -1\}$.

case 1 Assume that $\sigma_{ij} \notin I$ for some $i, j \in \{1, \dots, -1\}$ such that $i \neq j$. Set $Y := \{c \in C \mid c\sigma_{ij} \in I\}$. Since $\sigma_{ij} \notin I$, Y is a proper ideal of C . Hence it is contained in a maximal ideal m of C . Clearly $I \cap C \subseteq Y \subseteq m$ and hence $S_m \cap Y = \emptyset$. We show now that $\phi_m(g')$ does not commute with $\phi_m(h')$, i.e. $F_m(\sigma) \notin U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Obviously $(F_m(\sigma))_{ij} = f_m(\sigma_{ij})$. Assume $(F_m(\sigma))_{ij} \in I_m$. Then

$$\begin{aligned} & \exists x \in I, s \in S_m : \frac{\sigma_{ij}}{1} = \frac{x}{s} \\ \Rightarrow & \exists x \in I, s, t \in S_m : t(\sigma_{ij}s - x) = 0 \\ \Rightarrow & \exists x \in I, s, t \in S_m : st\sigma_{ij} = tx \in I \\ \Rightarrow & \exists u \in S_m : u\sigma_{ij} \in I. \end{aligned}$$

But this contradicts $S_m \cap Y = \emptyset$. Hence $(F_m(\sigma))_{ij} \notin I_m$ and thus $\phi_m(g')$ is noncentral.

case 2 Assume that $\sigma_{ii} - 1 \notin I$ for some $i \in \{1, \dots, -1\}$. Set $Y := \{c \in C \mid c(\sigma_{ii} - 1) \in I\}$. Since $\sigma_{ii} - 1 \notin I$, Y is a proper ideal of C . Hence it is contained in a maximal ideal m of C . Clearly $I \cap C \subseteq Y \subseteq m$ and hence $S_m \cap Y = \emptyset$. We show now that $\phi_m(g')$ does not commute with $\phi_m(h')$, i.e. $F_m(\sigma) \notin U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Obviously $(F_m(\sigma))_{ii} - 1 = f_m(\sigma_{ii}) - 1 = f_m(\sigma_{ii} - 1)$. Assume $(F_m(\sigma))_{ii} - 1 \in I_m$. Then

$$\begin{aligned} & \exists x \in I, s \in S_m : \frac{\sigma_{ii} - 1}{1} = \frac{x}{s} \\ \Rightarrow & \exists x \in I, s, t \in S_m : t((\sigma_{ii} - 1)s - x) = 0 \\ \Rightarrow & \exists x \in I, s, t \in S_m : st(\sigma_{ii} - 1) = tx \in I \\ \Rightarrow & \exists u \in S_m : u(\sigma_{ii} - 1) \in I. \end{aligned}$$

But this contradicts $S_m \cap Y = \emptyset$. Hence $(F_m(\sigma))_{ii} - 1 \notin I_m$ and thus $\phi_m(g')$ is noncentral.

case 3 Assume that $x_j = |\sigma_{*j}| \notin \Gamma$ for some $j \in \{1, \dots, -1\}$. Set $Y := \{c \in C \mid cx_j \in \Gamma\}$. Since $x_j \notin \Gamma$, Y is a proper ideal of C . Hence it is contained in a maximal ideal m of C . Since $x_j \in \Lambda$ and $y^2\Lambda \subseteq \Gamma_{\min} \subseteq \Gamma$ for any $y \in I \cap C$, $(I \cap C)^2 \subseteq Y \subseteq m$. This implies $S_m \cap Y = \emptyset$ and $I \cap C \subseteq m$, since m is prime. We show now that $\phi_m(g')$ does not commute with $\phi_m(h')$, i.e. $F_m(\sigma) \notin U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Obviously

$|(F_m(\sigma))_{*j}| = f_m(x_j)$. Assume $f_m(x_j) \in \Gamma_m$. Then

$$\begin{aligned} & \exists y \in \Gamma, s \in S_m : \frac{x_j}{1} = \frac{y}{s} \\ \Rightarrow & \exists y \in \Gamma, s, t \in S_m : t(x_j s - y) = 0 \\ \Rightarrow & \exists y \in \Gamma, s, t \in S : stx_j = ty \in \Gamma \\ \Rightarrow & \exists u \in S_m : ux_j \in \Gamma. \end{aligned}$$

But this contradicts $S_m \cap Y = \emptyset$. Hence $|(F_m(\sigma))_{*j}| = f_m(x_j) \notin \Gamma_m$ and thus $\phi_m(g')$ is noncentral. \square

Lemma 4.2 *Let (I, Γ) be a form ideal of (R, Λ) and m a maximal ideal of C . Then there is an $s_0 \in S_m$ with the properties*

(1) *if $x \in s_0 R$ and $\exists t \in S_m : tx \in I$, then $x \in I$ and*

(2) *if $x \in s_0 R$ and $\exists t \in S_m : tx \in \Gamma$, then $x \in \Gamma$.*

It follows that ϕ_m is injective on $\psi(U_{2n}((R, \Lambda), (s_0 R, s_0 \Lambda)))$.

Proof For any $s \in S_m$ set $Y(s) := \{x \in R | sx \in I\}$. Then for any $s \in S_m$, $Y(s)$ is a C -submodule of R . Since R is Noetherian C -module, the set $\{Y(s) | s \in S_m\}$ has a maximal element $Y(s_1)$. Clearly all elements $x \in s_1 R$ have the property that $tx \in I$ for some $t \in S_m$ implies $x \in I$. For any $s \in S$ set $Z(s) := \{x \in R | sx \in \Gamma\}$. Then for any $s \in S_m$, $Z(s)$ is a C -submodule of R . Since R is a Noetherian C -module, the set $\{Z(s) | s \in S\}$ has a maximal element $Z(s_2)$. Clearly all elements $x \in s_2 R$ have the property that $tx \in \Gamma$ for some $t \in S_m$ implies $x \in \Gamma$. Set $s_0 := s_1 s_2$. Since $s_0 R = s_1 s_2 R \subseteq s_1 R \cap s_2 R$, s_0 has the properties (1) and (2) above. We will show now that ϕ_m is injective on $\psi(U_{2n}((R, \Lambda), (s_0 R, s_0 \Lambda)))$. Let $g'_1, g'_2 \in \psi(U_{2n}((R, \Lambda), (s_0 R, s_0 \Lambda)))$ such that $\phi_m(g'_1) = \phi_m(g'_2)$. Since $g'_1, g'_2 \in \psi(U_{2n}((R, \Lambda), (s_0 R, s_0 \Lambda)))$, there are $g_1, g_2 \in U_{2n}((R, \Lambda), (s_0 R, s_0 \Lambda))$ such that $\psi(g_1) = g'_1$ and $\psi(g_2) = g'_2$. Set $h := (g_1)^{-1} g_2 \in U_{2n}((R, \Lambda), (s_0 R, s_0 \Lambda))$. Clearly $\phi_m(g'_1) = \phi_m(g'_2)$ is equivalent to $F_m(h) \in U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$, i.e.

(a) $F_m(h) \equiv e \pmod{I_m}$ and

(b) $f_m(|h_{*j}|) \in \Gamma_m \ \forall j \in \{1, \dots, -1\}$.

We want to show that $g'_1 = g'_2$ which is equivalent to $h \in U_{2n}((R, \Lambda), (I, \Gamma))$, i.e.

(a') $h \equiv e \pmod{I}$ and

(b') $x_j := |h_{*j}| \in \Gamma \ \forall j \in \{1, \dots, -1\}$.

First we show (a'). Let $i, j \in \{1, \dots, -1\}$ such that $i \neq j$. Since (a) holds, $f_m(h_{ij}) \in I_m$. Hence

$$\begin{aligned} & \exists x \in I, s \in S_m : \frac{h_{ij}}{1} = \frac{x}{s} \\ \Rightarrow & \exists x \in I, s, t \in S_m : t(h_{ij}s - x) = 0 \\ \Rightarrow & \exists x \in I, s, t \in S_m : sth_{ij} = tx \in I \\ \Rightarrow & \exists u \in S_m : uh_{ij} \in I. \end{aligned} \tag{4.1.1}$$

Since $h \in U_{2n}((R, \Lambda), (s_0R, s_0\Lambda))$, $h_{ij} \in s_0R$. It follows from (4.1.1) that $h_{ij} \in I$ since s_0 has property (1). Analogously one can show that $h_{ii} - 1 \in I$ for all $i \in \{1, \dots, -1\}$. Hence $h \equiv e \pmod{I}$. Now we show (b'). Let $j \in \{1, \dots, -1\}$. Since (b) holds, $f_m(x_j) \in \Gamma_m$. Hence

$$\begin{aligned} & \exists y \in \Gamma, s \in S_m : \frac{x_j}{1} = \frac{y}{s} \\ \Rightarrow & \exists y \in \Gamma, s, t \in S_m : t(x_js - y) = 0 \\ \Rightarrow & \exists y \in \Gamma, s, t \in S_m : stx_j = ty \in \Gamma \\ \Rightarrow & \exists u \in S_m : ux_j \in \Gamma. \end{aligned} \tag{4.1.2}$$

Since $h \in U_{2n}((R, \Lambda), (s_0R, s_0\Lambda))$, $x_j \in s_0\Lambda$. It follows from (4.1.2) that $x_j \in \Gamma$ since s_0 has property (2). Hence $g'_1 = g'_2$ and thus ϕ_m is injective on $\psi(U_{2n}((R, \Lambda), (s_0R, s_0\Lambda)))$. \square

Definition 4.3 Let G denote a group and A a set of subgroups of G such that

- (1) for any $U, V \in A$ there is a $W \in A$ such that $W \subseteq U \cap V$ and
- (2) for any $g \in G$ and $U \in A$ there is a $V \in A$ such that ${}^gV \subseteq U$.

Then A is called a *base of open subgroups* of $1 \in G$. A pair (A, B) is called a *supplemented base* for G if A and B are sets of nontrivial subgroups of G such that A is a base of open subgroups of $1 \in G$, each member of B is contained in some member of A , and if $U \in A$ and $V \in B$ then $U \cap V$ contains a member of B .

In the lemma below we use the following conventions. Let $x \in R$. Then RxR denotes the *involution invariant ideal generated by x* , i.e. the ideal of R generated by $\{x, \bar{x}\}$. Now let (I, Γ) be a form ideal of (R, Λ) and assume that $x \in R \setminus I$ or $x \in \Gamma_{max}^I \setminus \Gamma$. Set $\Gamma(x) := \Gamma_{min}^{RxR}$ if $x \in R \setminus I$ and $\Gamma(x) := \Gamma_{min}^{RxR} + \langle \{yxy | y \in R\} \rangle$ if $x \in \Gamma_{max}^I \setminus \Gamma$. $\Gamma(x)$ is called the *relative form parameter defined by x and (I, Γ)* . One checks easily that $(RxR, \Gamma(x))$ is a form ideal of (R, Λ) which is not contained in (I, Γ) , i.e. $RxR \not\subseteq I$ or $\Gamma(x) \not\subseteq \Gamma$. It is called the *form ideal defined by x and (I, Γ)* .

Lemma 4.4 Let (I, Γ) be a form ideal of (R, Λ) , m a maximal ideal of C and $s_0 \in S_m$ as in the previous lemma. Set $A := \{EU_{2n}(ss_0R, ss_0\Lambda) | s \in S_m\}$ and $B := \{EU_{2n}(Rxs_0R, \Gamma(xs_0)) | (x \in R, xs_0 \in R \setminus I) \vee (x \in \Lambda, xs_0 \in I \setminus \Gamma)\}$. Then (A, B) is a supplemented base for $EU_{2n}(R, \Lambda)$ and $F_m(A, B) := (F_m(A), F_m(B))$ is a supplemented base for $EU_{2n}(R_m, \Lambda_m)$.

Proof First we show (A, B) is a supplemented base for $EU_{2n}(R, \Lambda)$. Clearly A and B are sets of nontrivial subgroups of $EU_{2n}(R, \Lambda)$. We show now that A is a base of open subgroups of $1 \in EU_{2n}(R, \Lambda)$. Therefore we must show that A satisfies the conditions (1) and (2) in Definition 4.3.

- (1) Let $U = EU_{2n}(ss_0R, ss_0\Lambda), V = EU_{2n}(ts_0R, ts_0\Lambda) \in A$. Set $W := EU_{2n}(sts_0R, sts_0\Lambda) \in A$. Then clearly $W \subseteq U \cap V$.

- (2) Let $g \in EU_{2n}(R, \Lambda)$ and $U = EU_{2n}(ss_0R, ss_0\Lambda) \in A$. There is a $K \in \mathbb{N}$ such that g is the product of K elementary unitary matrices. Set $V := EU_{2n}((ss_0)^{2 \cdot 4^K + 4^{K-1} + \dots + 4}R, (ss_0)^{2 \cdot 4^K + 4^{K-1} + \dots + 4}\Lambda) \in A$. Then ${}^gV \subseteq U$ (see Lemma 4.1 in [5]).

Hence A is a base of open subgroups of $1 \in EU_{2n}(R, \Lambda)$. Let $EU_{2n}(Rxs_0R, \Gamma(xs_0)) \in B$. Then $EU_{2n}(Rxs_0R, \Gamma(xs_0)) \subseteq EU_{2n}(s_0R, s_0\Lambda) \in A$. It remains to show that if $U \in A$ and $V \in B$ then $U \cap V$ contains a member of B . Let $U = EU_{2n}(ss_0R, ss_0\Lambda) \in A$ and $V = EU_{2n}(Rxs_0R, \Gamma(xs_0)) \in B$. Set $W := EU_{2n}(Rxs_0R, \Gamma(xss_0))$. If $xs_0 \notin I$, then $xss_0 \notin I$ and if $xs_0 \notin \Gamma$, then $xss_0 \notin \Gamma$ (by the definition of s_0 , see the previous lemma). Hence $W \in B$. Obviously $W \in U \cap V$. Hence (A, B) is a supplemented base for $EU_{2n}(R, \Lambda)$.

Now we show $F_m(A, B)$ is a supplemented base for $EU_{2n}(R_m, \Lambda_m)$. Clearly $F_m(A)$ and $F_m(B)$ are sets of nontrivial subgroups of $EU_{2n}(R_m, \Lambda_m)$. We show now that $F_m(A)$ is a base of open subgroups of $1 \in EU_{2n}(R_m, \Lambda_m)$. Therefore we must show that $F_m(A)$ satisfies the conditions (1) and (2) in Definition 4.3.

- (1) Let $U = F_m(EU_{2n}(ss_0R, ss_0\Lambda))$, $V = F_m(EU_{2n}(ts_0R, ts_0\Lambda)) \in F_m(A)$. Set $W := F_m(EU_{2n}(sts_0R, sts_0\Lambda)) \in F_m(A)$. Then clearly $W \subseteq U \cap V$.
- (2) Let $g \in EU_{2n}(R_m, \Lambda_m)$ and $U = F_m(EU_{2n}(ts_0R, ts_0\Lambda)) \in F_m(A)$. There are a $K \in \mathbb{N}$ and elementary unitary matrices $\tau_1 = T_{i_1j_1}(\frac{x_1}{s_1}), \dots, \tau_K = T_{i_Kj_K}(\frac{x_K}{s_K}) \in EU_{2n}(R_m, \Lambda_m)$ such that $g = \tau_1 \dots \tau_K$. Set $s := s_1 \dots s_K$ and $V := F_m(EU_{2n}((sts_0)^{2 \cdot 4^K + 4^{K-1} + \dots + 4}R, (sts_0)^{2 \cdot 4^K + 4^{K-1} + \dots + 4}\Lambda)) \in F_m(A)$. Then ${}^gV \subseteq U$ (see Lemma 4.1 in [5]).

Hence $F_m(A)$ is a base of open subgroups of $1 \in EU_{2n}(R_m, \Lambda_m)$. That each member of $F_m(B)$ is contained in some member of $F_m(A)$ follows from the fact that any member of B is contained in a member of A . That given $U \in F_m(A)$ and $V \in F_m(B)$, $U \cap V$ contains a member of $F_m(B)$ follows from the fact that given $U \in A$ and $V \in B$, $U \cap V$ contains a member of B . Hence $F_m(A, B)$ is a supplemented base for $EU_{2n}(R_m, \Lambda_m)$. \square

The lemmas 4.5, 4.6 and 4.7 will be used in the proof of Lemma 4.8.

Lemma 4.5 *Let (I, Γ) be a form ideal of (R, Λ) and $S \subseteq C$ a multiplicative subset. Let $T_{ij}(x) \in EU_{2n}(R_m, \Lambda_m)$ be an elementary short or long root element, $\sigma \in U_{2n}(R_m, \Lambda_m)$ and $s \in S$. Then $[\sigma, T_{ij}(x)] \in U_{2n}((R_m, \Lambda_m), (I_m, \Lambda_m \cap I_m))$ if and only if $[\sigma, T_{ij}(f_m(s)x)] \in U_{2n}((R_m, \Lambda_m), (I_m, \Lambda_m \cap I_m))$.*

Proof Straightforward computation. \square

Lemma 4.6 *Let $\sigma \in U_{2n}(R, \Lambda)$. Further let $i, j \in \{1, \dots, -1\}$ such that $i \neq \pm j$, $x \in R$ and $y \in \lambda^{-(\epsilon(i)+1)/2}\Lambda$. Set $\tau := [\sigma, T_{ij}(x)]$ and $\rho := [\sigma, T_{i,-i}(y)]$. Then*

$$|\tau_{*k}| = \bar{\sigma}'_{jk}\bar{x}|\sigma_{*i}|x\sigma'_{jk} + \bar{\sigma}'_{-i,k}x|\sigma_{*, -j}|\bar{x}\sigma'_{-i,k} + a_k - \lambda\bar{a}_k,$$

if $k \neq j, -i$,

$$|\tau_{*j}| = \bar{\sigma}'_{jj}\bar{x}|\sigma_{*i}|x\sigma'_{jj} + \bar{\sigma}'_{-i,j}x|\sigma_{*, -j}|\bar{x}\sigma'_{-i,j} + \bar{x}|\tau_{*i}|x + a_j - \lambda\bar{a}_j$$

and

$$|\tau_{*, -i}| = \bar{\sigma}'_{j, -i} \bar{x} |\sigma_{*i}| x \sigma'_{j, -i} + \bar{\sigma}'_{-i, -i} x |\sigma_{*, -j}| \bar{x} \sigma'_{-i, -i} + x |\tau_{*, -j}| \bar{x} + a_{-i} - \lambda \bar{a}_{-i}$$

where for each $k \in \{1, \dots, -1\}$, a_k lies in the ideal $J(\sigma)$ generated by the nondiagonal entries of σ and σ^{-1} . It follows that if I is an involution invariant ideal and $\sigma \in U_{2n}((R, \Lambda), (I, I \cap \Lambda))$, then $|\tau_{*k}| \in \Gamma_{min}^I \ \forall k \neq j, -i$, $|\tau_{*j}| \equiv \bar{x} |\sigma_{*i}| x \pmod{\Gamma_{min}^I}$ and $|\tau_{*, -i}| \equiv x |\sigma_{*, -j}| \bar{x} \pmod{\Gamma_{min}^I}$. Further

$$|\rho_{*k}| = \bar{\sigma}'_{-i, k} \bar{y} |\sigma_{*i}| y \sigma'_{-i, k} + b_k,$$

if $k \neq -i$,

$$|\rho_{*, -i}| = \bar{\sigma}'_{-i, -i} \bar{y} |\sigma_{*i}| y \sigma'_{-i, -i} + \bar{y} |\rho_{*i}| y + b_{-i} + c - \lambda \bar{c}$$

where

$$b_k = \begin{cases} \sigma_{-k, i} y \sigma'_{-i, k}, & \text{if } k \neq -i \text{ and } \epsilon(k) = 1, \\ \sigma_{-k, i} y \sigma'_{-i, k}, & \text{if } k \neq -i \text{ and } \epsilon(k) = -1, \\ \sigma_{ii} y \sigma'_{-i, i} - y, & \text{if } k = -i \text{ and } \epsilon(i) = 1, \\ \sigma_{ii} y \sigma'_{-i, -i} - y, & \text{if } k = -i \text{ and } \epsilon(i) = -1, \end{cases}$$

and $c \in J(\sigma)$. It follows that if I is an involution invariant ideal and $\sigma \in U_{2n}((R, \Lambda), (I, I \cap \Lambda))$, then $|\rho_{*k}| \in \Gamma_{min}^I \ \forall k \neq -i$ and $|\rho_{*, -i}| \equiv \bar{y} |\sigma_{*i}| y \pmod{\Gamma_{min}^I}$.

Proof Straightforward computation. \square

Lemma 4.7 Let m be a maximal ideal of C and $\sigma \in U_{2n}(R_m, \Lambda_m)$. Then there is an $\epsilon \in EU_{2n}(R_m, \Lambda_m)$ such that $(\epsilon\sigma)_{11}$ is invertible.

Proof By Lemma 1.4 in [7] and Lemma 3.4 in [3], R_m satisfies the Λ -stable range condition ΛS_1 . Hence there is an $\epsilon_1 = \begin{pmatrix} e^{n \times n} & 0 \\ \gamma & e^{n \times n} \end{pmatrix} \in EU_{2n}(R_m, \Lambda_m)$, where $\gamma \in M_n(R_m)$, such that (x_1, \dots, x_n) is right unimodular where (x_1, \dots, x_{-1}) is the first row of ${}^{\epsilon_1}\sigma$. Since ΛS_1 implies $S R_1$, there is a matrix $\epsilon_2 = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} \in EU_{2n}(R_m, \Lambda_m)$, where ω_1 and ω_2 are lower triangular matrices in $M_n(R_m)$ with 1's on the diagonal, such that the entry of $({}^{\epsilon_2\epsilon_1}\sigma)_{11}$ is right invertible. Since R is a Noetherian C -module, R is almost commutative. It follows that R_m is almost commutative and hence $({}^{\epsilon_2\epsilon_1}\sigma)_{11}$ is invertible (note that almost commutative rings are Dedekind-finite by Nakayama's Lemma). \square

In the following lemma we will apply lemmas and corollaries in [2], chapter IV, §3. We are allowed to do this since for any maximal ideal m of C , $C'_m := S_m^{-1}C'$ is semilocal by Lemma 1.4 in [7] and hence the Bass-Serre-dimension of C'_m is 0. Since R is module finite over C' , R_m is module finite over C'_m and hence R_m is a finite C'_m -algebra. Further set $A' := \phi_m(\psi(A))$ and $B' := \phi_m(\psi(B))$ where (A, B) is the supplemented base for $EU_{2n}(R, \Lambda)$ defined in Lemma 4.4. Clearly (A', B') is a supplemented base for $\psi_m(EU_{2n}(R_m, \Lambda_m))$ since $F_m(A, B)$ is a supplemented base for $EU_{2n}(R_m, \Lambda_m)$ by Lemma 4.4 and $\phi_m \circ \psi = \psi_m \circ F_m$.

Lemma 4.8 Let (I, Γ) be a form ideal of (R, Λ) , m a maximal ideal of C and $h' \in U_{2n}(R_m, \Lambda_m)/U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$ a noncentral element. Then given any $U' \in A'$, there is a $k \in \mathbb{N}$ and elements $g'_0, \dots, g'_k \in \phi_m(\psi(EU_{2n}(R, \Lambda)))$, $\epsilon'_0, \dots, \epsilon'_k \in \psi_m(EU_{2n}(R_m, \Lambda_m))$ and $l_1, \dots, l_k \in \{-1, 1\}$ such that g'_k is the nontrivial image of an elementary matrix in $EU_{2n}(R, \Lambda)$,

$$\epsilon'_k([\epsilon'_{k-1}(\dots \epsilon'_2([\epsilon'_1([\epsilon'_0 h', g'_0]^{l_1}), g'_1]^{l_2}) \dots), g'_{k-1}]^{l_k}) = g'_k$$

and

$$d'_i g'_i \in U' \quad \forall i \in \{0, \dots, k\}$$

where $d'_i = (\epsilon'_i \cdot \dots \cdot \epsilon'_0)^{-1}$ ($0 \leq i \leq k$).

Proof Let $U' \in A'$. Then there is a $U \in F_m(A)$ such that $\psi_m(U) = U'$. Let $h \in U_{2n}(R_m, \Lambda_m)$ such that $h' = \psi_m(h)$. Since h' is noncentral, $h \notin CU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. The proof is divided into three parts, I, II and III. In Part I we assume that $h \notin CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ and $n > 3$. In Part II we assume that $h \notin CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ and $n = 3$. In Part III we assume that $h \in CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. The groups $U_i \in F_m(A)$ ($0 \leq i \leq k$) appearing in the proof are chosen such that $d_i U_i \subseteq U$ where $d_i = (\epsilon_i \cdot \dots \cdot \epsilon_0)^{-1}$ (possible by Lemma 4.4). The elements $t_i \in S_m$ ($0 \leq i \leq k$) are chosen such that $U_i = F_m(EU_{2n}(t_i s_0 R, t_i s_0 \Lambda))$. Further we denote $f_m(t_i s_0)$ by s_i ($0 \leq i \leq k$).

Part I Assume that $h \notin CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ and $n > 3$.

By [2], chapter IV, Corollary 3.10 (applied with $H = {}^{EU_{2n}(R_m, \Lambda_m)}\langle h \rangle$), there is an $\epsilon_0 \in EU_{2n}(R_m, \Lambda_m)$ and an $x = \frac{a}{s} \in R_m$ such that $[\epsilon_0 h, T_{1,-2}(x)] \notin CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. By [2], chapter IV, Lemma 3.12, Part I, case 7 there is a matrix $\epsilon_1 \in EU_{2n}(R_m, \Lambda_m)$ of the form

$$\epsilon_1 = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix},$$

where $X, Y, Z \in M_n(R_m)$, such that the first n coordinates of $\epsilon_1({}^{\epsilon_0}h)_{*1}$ equal $(1 \ 0 \ \dots \ 0)^t$ and the first n coordinates of $\epsilon_1({}^{\epsilon_0}h)_{*2}$ equal $(0 \ 1 \ 0 \ \dots \ 0)^t$. Set $f_m(a) := \hat{a}$ and $f_m(s) := \hat{s}$. Set $g_0 := T_{1,-2}(s_0 \hat{s}x) = T_{1,-2}(s_0 \frac{s}{1} \frac{a}{s}) = T_{1,-2}(s_0 \hat{a}) \in U_0$. By Lemma 4.5, $[\epsilon_0 h, g_0] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Since $U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ is normal, it follows that $\sigma := \epsilon_1[\epsilon_0 h, g_0] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Since

$$\begin{aligned} & \sigma \\ &= \epsilon_1[\epsilon_0 h, g_0] \\ &= \epsilon_1(g_0^{-1} + ({}^{\epsilon_0}h)_{*1} s_0 \hat{a} (({}^{\epsilon_0}h)^{-1})_{-2,*} g_0^{-1} \\ & \quad - ({}^{\epsilon_0}h)_{*2} \bar{\lambda}_m \overline{s_0 \hat{a}} (({}^{\epsilon_0}h)^{-1})_{-1,*} g_0^{-1}) \\ &= \epsilon_1(g_0^{-1}) + \epsilon_1(({}^{\epsilon_0}h)_{*1} s_0 \hat{a} (({}^{\epsilon_0}h)^{-1})_{-2,*} g_0^{-1}) \\ & \quad - \epsilon_1(({}^{\epsilon_0}h)_{*2} \bar{\lambda}_m \overline{s_0 \hat{a}} (({}^{\epsilon_0}h)^{-1})_{-1,*} g_0^{-1}) \\ &= \epsilon_1(g_0^{-1}) + \epsilon_1({}^{\epsilon_0}h)_{*1} s_0 \hat{a} (({}^{\epsilon_0}h)^{-1})_{-2,*} g_0^{-1} (\epsilon_1)^{-1} \\ & \quad - \epsilon_1({}^{\epsilon_0}h)_{*2} \bar{\lambda}_m \overline{s_0 \hat{a}} (({}^{\epsilon_0}h)^{-1})_{-1,*} g_0^{-1} (\epsilon_1)^{-1} \end{aligned}$$

and ${}^{\epsilon_1}(g_0^{-1}) = e + (\epsilon_1)_{*1} s_0 \hat{a}((\epsilon_1)^{-1})_{-2,*} - (\epsilon_1)_{*2} \bar{\lambda}_m \overline{s_0 \hat{a}}((\epsilon_1)^{-1})_{-1,*}$ has the form

$$\begin{pmatrix} e^{n \times n} & * \\ 0 & e^{n \times n} \end{pmatrix},$$

σ has the form

$$\left(\begin{array}{cc|cc} \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \\ 0 & e^{2 \times 2} & \beta_3 & \beta_4 \\ \hline & \gamma & & \delta \end{array} \right) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where $\alpha_1, \beta_2 \in M_{n-2}(R_m)$, $\alpha_2, \beta_1 \in M_{(n-2) \times 2}(R_m)$, $\beta_3 \in M_2(R_m)$, $\beta_4 \in M_{2 \times (n-2)}(R_m)$ and $\alpha = (a_{ij})_{1 \leq i, j \leq n}$, $\beta = (b_{ij})_{\substack{1 \leq i \leq n \\ -n \leq j \leq -1}}$, $\gamma = (c_{ij})_{\substack{-n \leq i \leq -1 \\ 1 \leq j \leq n}}$, $\delta = (d_{ij})_{-n \leq i, j \leq -1} \in M_n(R_m)$.

case 1 Assume that either $\alpha \not\equiv e^{n \times n}(\text{mod } I_m)$ or $\gamma \not\equiv 0(\text{mod } I_m)$ or $\delta \not\equiv e^{n \times n}(\text{mod } I_m)$.

We will show that it follows that $\alpha \not\equiv e^{n \times n}(\text{mod } I_m)$ or $\gamma \not\equiv 0(\text{mod } I_m)$. Assume that $\alpha \equiv e^{n \times n}(\text{mod } I_m)$ and $\gamma \equiv 0(\text{mod } I_m)$. Let $\kappa : M_n(R_m) \rightarrow M_n(R_m/I_m)$ be the homomorphism induced by the canonical homomorphism $R_m \rightarrow R_m/I_m$. Since the image of σ in $U_{2n}(R_m/I_m, \Lambda_m/(\Lambda_m \cap I_m))$ equals $\begin{pmatrix} e^{n \times n} & \kappa(\beta) \\ 0 & \kappa(\delta) \end{pmatrix}$, $\kappa(\delta) = e^{n \times n}$ by Lemma 3.9. That is equivalent to $\delta \equiv e^{n \times n}(\text{mod } I_m)$. Since this is a contradiction, $\alpha \not\equiv e^{n \times n}(\text{mod } I_m)$ or $\gamma \not\equiv 0(\text{mod } I_m)$. Hence there is an $i \in \{1, \dots, n\}$ such that $\sigma_{*i} \not\equiv e_i(\text{mod } I_m)$.

case 1.1 Assume that $i \in \{1, \dots, n-2\}$.

Clearly the $(n-1)$ -th row of

$$\begin{aligned} & [\sigma, T_{i, -(n-1)}(1)] \\ &= (e + \sigma_{*i} \sigma'_{-(n-1),*} - \bar{\lambda}_m \sigma_{*, n-1} \sigma'_{-i,*}) T_{i, -(n-1)}(-1) \\ &= (e + \begin{matrix} 1 \\ n \\ -n \\ -1 \end{matrix} \begin{pmatrix} a_{1i} \\ \vdots \\ a_{n-2,i} \\ 0 \\ 0 \\ c_{-n,i} \\ \vdots \\ c_{-1,i} \end{pmatrix} \begin{pmatrix} & & & & \\ & 1 & n & -n & -1 \\ \lambda_m \bar{c}_{-1, n-1} \dots \lambda_m \bar{c}_{-n, n-1} & 0 & 1 & \bar{a}_{n-2, n-1} \dots \bar{a}_{1, n-1} \end{pmatrix}) \end{aligned}$$

$$\begin{aligned}
& \begin{pmatrix} 1 & a_{1,n-1} \\ & \vdots \\ & a_{n-2,n-1} \\ -\bar{\lambda}_m & 1 \\ n & 0 \\ & c_{-n,n-1} \\ -n & \vdots \\ & c_{-1,n-1} \\ -1 & \end{pmatrix} \begin{pmatrix} 1 & n & -n & -1 \\ \lambda_m \bar{c}_{-1,i} \dots \lambda_m \bar{c}_{-n,i} & 0 & 0 & \bar{a}_{n-2,i} \dots \bar{a}_{1i} \end{pmatrix} \\
& \cdot T_{i,-(n-1)}(-1)
\end{aligned}$$

is not congruent to f_{n-1} modulo I_m since $\sigma_{*i} \not\equiv e_i \pmod{I_m}$. Recall that $f_l = e_l^t$ for any $l \in \{1, \dots, -1\}$. Hence $[\sigma, T_{i,-(n-1)}(1)] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Set $g_1 := T_{i,-(n-1)}(s_1) \in U_1$. By Lemma 4.5, $[\sigma, g_1] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Clearly the n -th row of $[\sigma, g_1]$ equals f_n . Let P_{1n} be as in 3.11. Set $\epsilon_2 := P_{1n} \in EU_{2n}(R_m, \Lambda_m)$. Then the first row of $\tau := {}^{\epsilon_2}[\sigma, g_1]$ equals f_1 . Since $U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ is normal, $\tau \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Clearly τ has the form

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ A_3 & A_4 & B_3 & 0 \\ \hline C_1 & C_2 & D_1 & 0 \\ C_3 & C_4 & D_3 & 1 \end{array} \right) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $C_3 \in R_m$, $A_3, C_1 \in (R_m)^{n-1}$, $C_4, D_3 \in {}^{n-1}(R_m)$, $A_4, B_3, C_2, D_1 \in M_{n-1}(R_m)$ and $A = (A_{ij})_{1 \leq i, j \leq n}$, $B = (B_{ij})_{\substack{1 \leq i \leq n \\ -n \leq j \leq -1}}$, $C = (C_{ij})_{\substack{-n \leq i \leq -1 \\ 1 \leq j \leq n}}$, $D = (D_{ij})_{-n \leq i, j \leq -1} \in M_n(R_m)$. Set

$$E = (E_{ij})_{2 \leq i, j \leq -2} := \left(\begin{array}{c|c} A_4 & B_3 \\ \hline C_2 & D_1 \end{array} \right) \in M_{2n-2}(R_m).$$

case 1.1.1 Assume that $E \not\equiv e^{(2n-2) \times (2n-2)} \pmod{I_m}$.

There are $i, j \in \{2, \dots, -2\}$ such that $(E - e^{(2n-2) \times (2n-2)})_{ij} \notin I_m$. Set $g_2 := T_{-1,i}(s_2) \in U_2$. Then $\omega := [\tau^{-1}, g_2]$ has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ * & e^{(2n-2) \times (2n-2)} & 0 \\ * & w & 1 \end{pmatrix}$$

where $w = (w_2, \dots, w_{-2}) = s_2(E - e^{(2n-2) \times (2n-2)})_{i*}$. Since $(E - e^{(2n-2) \times (2n-2)})_{ij} \notin I_m$, $w_j =: \frac{b'}{t'} \notin I_m$. Set $b := \frac{b'}{1} \in R_m$ and $t := \frac{t'}{1} \in R_m$. Choose an $l \neq \pm 1, \pm j$ and set $g_3 := T_{jl}(s_3) \in U_3$, $g_4 := T_{l,-j}(s_4 s_5 t) \in U_4$ and $g_5 := T_{-1,-j}(s_3 s_4 s_5 t w_j) = T_{-1,-j}(s_3 s_4 s_5 b) \in U_5$. Notice that $g_5 \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, since $w_j \notin I_m$

and $s_3 s_4 s_5 t$ is invertible. One checks easily that

$$\begin{aligned} & [[[\epsilon^2([\epsilon^1([\epsilon^0 h, g_0], g_2)^{-1}), g_2], g_3], g_4] \\ &= [[[(\epsilon^2([\epsilon^1([\epsilon^0 h, g_0], g_2))^{-1}, g_2], g_3], g_4] \\ &= [[\omega, g_3], g_4] \\ &= g_5. \end{aligned}$$

Set $g'_i := \psi_m(g_i) \ \forall i \in \{0, \dots, 5\}$ and $\epsilon'_i := \psi_m(\epsilon_i) \ \forall i \in \{0, 1, 2\}$. Then

$$[[[\epsilon'_2([\epsilon'_1([\epsilon'_0 h', g'_0], g'_1)^{-1}), g'_2], g'_3], g'_4] = g'_5.$$

case 1.1.2 Assume that $E \equiv e^{(2n-2) \times (2n-2)} \pmod{I_m}$ and $A_3 \equiv 0 \pmod{I_m}$.

Set $\xi_1 := \prod_{l=2}^n T_{l1}(-A_{l1}) \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Since $\xi_1 \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m)) \subseteq U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m)) \subseteq U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ and $\tau \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, $\xi_1 \tau \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Clearly

$$\xi_1 \tau = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & A_4 & B_3 & 0 \\ \hline C_1 & C_2 & D_1 & 0 \\ C'_3 & C'_4 & D'_3 & 1 \end{array} \right)$$

for some $C'_3 \in R_m$ and $C'_4, D'_3 \in {}^{n-1}(R_m)$ such that $D'_3 \equiv 0 \pmod{I_m}$ (consider the image of $\xi_1 \tau$ in $U_{2n}(R_m/I_m, \Lambda_m/(\Lambda_m \cap I_m))$).

case 1.1.2.1 Assume that there is an $i \in \{3, \dots, n\}$ such that $C'_{-1,i} \notin I_m$.

Set $\epsilon_{21} := T_{12}(-1) \in EU_{2n}(R_m, \Lambda_m)$. Then ${}^{\epsilon_{21}}(\xi_1 \tau)$ has the form

$$\left(\begin{array}{cc|cc} 1 & A''_2 & B''_1 & B''_2 \\ 0 & A_4 & B''_3 & B''_4 \\ \hline C''_1 & C''_2 & D''_1 & D''_2 \\ C''_3 & C''_4 & D''_3 & D''_4 \end{array} \right)$$

where $B''_2, C''_3, D''_4 \in R_m$, $B''_4, C''_1 \in (R_m)^{n-1}$, $A''_2, B''_1, C''_4, D''_3 \in {}^{n-1}(R_m)$, $B''_3, C''_2, D''_1 \in M_{n-1}(R_m)$. Furthermore $A''_2 \equiv 0 \pmod{I_m}$ and $C''_{-2,i} \equiv C'_{-1,i} \pmod{I_m}$. Set $\xi_2 := \prod_{l=2}^n T_{1l}(-A''_{1l}) \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. $\omega := {}^{\epsilon_{21}}(\xi_1 \tau) \xi_2$ has the form

$$\left(\begin{array}{cc|cc} 1 & 0 & B'''_1 & B'''_2 \\ 0 & A_4 & B'''_3 & B'''_4 \\ \hline C'''_1 & C'''_2 & D'''_1 & D'''_2 \\ C'''_3 & C'''_4 & D'''_3 & D'''_4 \end{array} \right)$$

where $B'''_2, C'''_3, D'''_4 \in R_m$, $B'''_4, C'''_1, D'''_2 \in (R_m)^{n-1}$, $B'''_1, C'''_4, D'''_3 \in {}^{n-1}(R_m)$, $B'''_3, C'''_2, D'''_1 \in M_{n-1}(R_m)$. Further $C'''_{-2,i} \equiv C''_{-2,i} \equiv C'_{-1,i} \pmod{I_m}$. Since $C'_{-1,i} \notin I_m$,

$C'''_{-2,i} \notin I_m$. Set $g_2 := T_{2,-i}(s_2) \in U_2$. Then

$$\begin{aligned}
& [\omega, g_2] \\
& = (e + \omega_{*2} s_2 \omega'_{-i,*} - \omega_{*i} \bar{\lambda}_m s_2 \omega'_{-2,*}) (g_2)^{-1} \\
& = (e + s_2 \begin{pmatrix} 0 \\ A_{22} \\ \vdots \\ A_{n2} \\ C'''_{-n,2} \\ \vdots \\ C'''_{-1,2} \end{pmatrix} \begin{pmatrix} 1 & n & -n & -1 \\ \lambda_m \bar{C}'''_{-1,i} & \dots & \lambda_m \bar{C}'''_{-n,i} & \bar{A}_{ni} & \dots & \bar{A}_{2i} & 0 \end{pmatrix} \\
& \quad - \bar{\lambda}_m s_2 \begin{pmatrix} 0 \\ A_{2i} \\ \vdots \\ A_{ni} \\ C'''_{-n,i} \\ \vdots \\ C'''_{-1,i} \end{pmatrix} \begin{pmatrix} 1 & n & -n & -1 \\ \lambda_m \bar{C}'''_{-1,2} & \dots & \lambda_m \bar{C}'''_{-n,2} & \bar{A}_{n2} & \dots & \bar{A}_{22} & 0 \end{pmatrix}) \\
& \quad \cdot (g_2)^{-1}.
\end{aligned}$$

Clearly $[\omega, g_2]_{1*} = f_1$ and

$$[\omega, g_2]_{22} = 1 + s_2 A_{22} \lambda_m \bar{C}'''_{-2,i} - \bar{\lambda}_m s_2 A_{2i} \lambda_m \bar{C}'''_{-2,2}.$$

Since $A_{2i} \in I_m$, $-\bar{\lambda}_m s_2 A_{2i} \lambda_m \bar{C}'''_{-2,2} \in I_m$. Since $C'''_{-2,i} \notin I_m$, $\bar{C}'''_{-2,i} \notin I_m$. Hence $s_2 A_{22} \lambda_m \bar{C}'''_{-2,i} \notin I_m$ since $A_{22} \equiv 1 \pmod{I_m}$ and s_2 and λ_m are invertible. It follows that $[\omega, g_2]_{22} \not\equiv 1 \pmod{I_m}$. One can proceed now as in case 1.1.1 (note that $\psi_m(\xi_1) = \psi_m(\xi_2) = e$ since $\xi_1, \xi_2 \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m)) \subseteq U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$).

case 1.1.2.2 Assume that $C'_{-1,2} \notin I_m$.

Set $\epsilon_{21} := T_{13}(-1) \in EU_{2n}(R_m, \Lambda_m)$. Then $\epsilon_{21}(\xi_1 \tau)$ has the form

$$\left(\begin{array}{cc|cc} 1 & A''_2 & B''_1 & B''_2 \\ 0 & A_4 & B''_3 & B''_4 \\ \hline C''_1 & C''_2 & D''_1 & D''_2 \\ C''_3 & C''_4 & D''_3 & D''_4 \end{array} \right)$$

where $B''_2, C''_3, D''_4 \in R_m$, $B''_4, C''_1 \in (R_m)^{n-1}$, $A''_2, B''_1, C''_4, D''_3 \in {}^{n-1}(R_m)$, $B''_3, C''_2, D''_1 \in M_{n-1}(R_m)$. Furthermore $A''_2 \equiv 0 \pmod{I_m}$ and $C'''_{-3,2} \equiv C'_{-1,2} \pmod{I_m}$. Set $\xi_2 :=$

$\prod_{l=2}^n T_{1l}(-A''_{1l}) \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Then $\omega := {}^{\epsilon_{21}}(\xi_1 \tau) \xi_2$ has the form

$$\left(\begin{array}{cc|cc} 1 & 0 & B_1''' & B_2''' \\ 0 & A_4 & B_3''' & B_4''' \\ \hline C_1''' & C_2''' & D_1''' & D_2''' \\ C_3''' & C_4''' & D_3''' & D_4''' \end{array} \right)$$

where $B_2''', C_3''', D_4''' \in R_m$, $B_4''', C_1''', D_2''' \in (R_m)^{n-1}$, $B_1''', C_4''', D_3''' \in {}^{n-1}(R_m)$, $B_3''', C_2''', D_1''' \in M_{n-1}(R_m)$. Further $C_{-3,2}''' \equiv C_{-3,2}'' \equiv C'_{-1,2} \pmod{I_m}$. Since $C'_{-1,2} \notin I_m$, $C_{-3,2}''' \notin I_m$. Set $g_2 := T_{3,-2}(s_2) \in U_2$. Then

$$\begin{aligned} & [\omega, g_2] \\ &= (e + \omega_{*3} s_2 \omega'_{-2,*} - \omega_{*2} \bar{\lambda}_m s_2 \omega'_{-3,*})(g_2)^{-1} \\ &= (e + s_2 \begin{pmatrix} 1 & 0 \\ n & A_{23} \\ -n & \vdots \\ -1 & A_{n3} \\ & C_{-n,3}''' \\ & \vdots \\ & C_{-1,3}''' \end{pmatrix} \begin{pmatrix} 1 & n & -n & -1 \\ \lambda_m \bar{C}_{-1,2}''' & \dots & \lambda_m \bar{C}_{-n,2}''' & \bar{A}_{n2} & \dots & \bar{A}_{22} & 0 \end{pmatrix}) \\ & \quad - \bar{\lambda}_m s_2 \begin{pmatrix} 1 & 0 \\ n & A_{22} \\ -n & \vdots \\ -1 & A_{n2} \\ & C_{-n,2}''' \\ & \vdots \\ & C_{-1,2}''' \end{pmatrix} \begin{pmatrix} 1 & n & -n & -1 \\ \lambda_m \bar{C}_{-1,3}''' & \dots & \lambda_m \bar{C}_{-n,3}''' & \bar{A}_{n3} & \dots & \bar{A}_{23} & 0 \end{pmatrix}) \\ & \quad \cdot (g_2)^{-1}. \end{aligned}$$

Clearly $[\omega, g_2]_{1*} = f_1$ and

$$[\omega, g_2]_{33} = 1 + s_2 A_{33} \lambda_m \bar{C}_{-3,2}''' - \bar{\lambda}_m s_2 A_{32} \lambda_m \bar{C}_{-3,3}'''.$$

Since $A_{32} \in I_m$, $-\bar{\lambda}_m s_2 A_{32} \lambda_m \bar{C}_{-3,3}''' \in I_m$. Since $C_{-3,2}''' \notin I_m$, $\bar{C}_{-3,2}''' \notin I_m$. Hence $s_2 A_{33} \lambda_m \bar{C}_{-3,2}''' \notin I_m$ since $A_{33} \equiv 1 \pmod{I_m}$ and s_2 and λ_m are invertible. It follows that $[\omega, g_2]_{33} \not\equiv 1 \pmod{I_m}$. One can proceed now as in case 1.1.1

case 1.1.2.3 Assume that $C'_{-1,i} \in I_m \ \forall i \in \{2, \dots, n\}$.

It follows that $C_{i1} \in I_m \forall i \in \{-n, \dots, -2\}$. Since $\xi_1\tau \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, $C'_{-1,1} \notin I_m$. Set $\epsilon_{21} := T_{12}(-1) \in EU_{2n}(R_m, \Lambda_m)$. Then ${}^{\epsilon_{21}}(\xi_1\tau)$ has the form

$$\left(\begin{array}{cc|cc} 1 & A_2'' & B_1'' & B_2'' \\ 0 & A_4 & B_3'' & B_4'' \\ \hline C_1'' & C_2'' & D_1'' & D_2'' \\ C_3'' & C_4'' & D_3'' & D_4'' \end{array} \right)$$

where $B_2'', C_3'', D_4'' \in R_m$, $B_4'', C_1'' \in (R_m)^{n-1}$, $A_2'', B_1'', C_4'', D_3'' \in {}^{n-1}(R_m)$, $B_3'', C_2'', D_1'' \in M_{n-1}(R_m)$. Furthermore $A_2'' \equiv 0 \pmod{I_m}$ and $C_{-2,2}'' = C_{-2,2} + C'_{-1,2} + C_{-2,1} + C'_{-1,1} \equiv C'_{-1,1} \pmod{I_m}$. Set $\xi_2 := \prod_{l=2}^n T_{1l}(-A_{1l}'') \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Then $\omega := {}^{\epsilon_{21}}(\xi_1\tau)\xi_2$ has the form

$$\left(\begin{array}{cc|cc} 1 & 0 & B_1''' & B_2''' \\ 0 & A_4 & B_3''' & B_4''' \\ \hline C_1''' & C_2''' & D_1''' & D_2''' \\ C_3''' & C_4''' & D_3''' & D_4''' \end{array} \right)$$

where $B_2''', C_3''', D_4''' \in R_m$, $B_4''', C_1''' \in (R_m)^{n-1}$, $B_1''', C_4''', D_3''' \in {}^{n-1}(R_m)$, $B_3''', C_2''', D_1''' \in M_{n-1}(R_m)$. Further $C_{-2,2}''' \equiv C_{-2,2}'' \equiv C'_{-1,1} \pmod{I_m}$. Since $C'_{-1,1} \notin I_m$, $C_{-2,2}''' \notin I_m$. Set $g_2 := T_{2,-3}(f_m(t_3 s_0)) \in U_2$. Then

$$\begin{aligned} & [\omega, g_2] \\ &= (e + \omega_{*2} s_2 \omega'_{-3,*} - \omega_{*3} \bar{\lambda}_m s_2 \omega'_{-2,*})(g_2)^{-1} \\ &= (e + s_2 \begin{pmatrix} 1 & 0 \\ n & A_{22} \\ -n & \vdots \\ -1 & A_{n2} \\ & C_{-n,2}''' \\ & \vdots \\ & C_{-1,2}''' \end{pmatrix} \begin{pmatrix} 1 & n & -n & -1 \\ \lambda_m \bar{C}_{-1,3}''' & \dots & \lambda_m \bar{C}_{-n,3}''' & \bar{A}_{n3} & \dots & \bar{A}_{23} & 0 \end{pmatrix} \\ & \quad - \bar{\lambda}_m s_2 \begin{pmatrix} 1 & 0 \\ n & A_{23} \\ -n & \vdots \\ -1 & A_{n3} \\ & C_{-n,3}''' \\ & \vdots \\ & C_{-1,3}''' \end{pmatrix} \begin{pmatrix} 1 & n & -n & -1 \\ \lambda_m \bar{C}_{-1,2}''' & \dots & \lambda_m \bar{C}_{-n,2}''' & \bar{A}_{n2} & \dots & \bar{A}_{22} & 0 \end{pmatrix}) \\ & \quad \cdot (g_2)^{-1}. \end{aligned}$$

Clearly $[\omega, g_2]_{1*} = f_1$ and

$$[\omega, g_2]_{32} = s_2 A_{32} \lambda_m \bar{C}_{-2,3}''' - \bar{\lambda}_m s_2 A_{33} \lambda_m \bar{C}_{-2,2}'''.$$

Since $A_{32} \in I_m$, $-\bar{\lambda}_m s_2 A_{32} \lambda_m \bar{C}_{-2,2}''' \in I_m$. Since $C_{-2,2}''' \notin I_m$, $\bar{C}_{-2,2}''' \notin I_m$. Hence $s_2 A_{33} \lambda_m \bar{C}_{-2,i}''' \notin I_m$ since $A_{33} \equiv 1 \pmod{I_m}$ and s_2 and λ_m are invertible. It follows that $[\omega, g_2]_{32} \notin I_m$. One can proceed now as in case 1.1.1

case 1.1.3 Assume that $E \equiv e^{(2n-2) \times (2n-2)} \pmod{I_m}$ and $A_3 \not\equiv 0 \pmod{I_m}$.

Since $A_3 \not\equiv 0 \pmod{I_m}$, $D_3 \not\equiv 0 \pmod{I_m}$. Hence there is an $i \in \{-n, \dots, -2\}$ such that $D_{-1,i} \notin I_m$. Choose a $j \in \{2, \dots, n\} \setminus \{-i\}$ and set $g_2 := T_{ij}(s_2) \in U_2$. Then

$$\begin{aligned} & [\tau, g_2] \\ &= (e + \tau_{*i} s_2 \tau'_{j*} - \tau_{*, -j} \lambda_m s_2 \tau'_{-i,*})(g_2)^{-1} \\ &= (e + s_2 \begin{pmatrix} 0 \\ B_{2i} \\ \vdots \\ B_{ni} \\ D_{-n,i} \\ \vdots \\ D_{-1,i} \end{pmatrix} \begin{pmatrix} 1 & n & -n & -1 \\ \bar{D}_{-1,-j} & \dots & \bar{D}_{-n,-j} & \bar{\lambda}_m \bar{B}_{n,-j} & \dots & \bar{\lambda}_m \bar{B}_{2,-j} & 0 \end{pmatrix} \\ & \quad - \lambda_m s_2 \begin{pmatrix} 0 \\ B_{2,-j} \\ \vdots \\ B_{n,-j} \\ D_{-n,-j} \\ \vdots \\ D_{-1,-j} \end{pmatrix} \begin{pmatrix} 1 & n & -n & -1 \\ \bar{D}_{-1,i} & \dots & \bar{D}_{-n,i} & \bar{\lambda}_m \bar{B}_{ni} & \dots & \bar{\lambda}_m \bar{B}_{2i} & 0 \end{pmatrix}) \\ & \quad \cdot (g_2)^{-1}. \end{aligned}$$

Clearly $[\tau, g_2]_{1*} = f_1$ and

$$\begin{aligned} & [\tau, g_2]_{-1,j} \\ &= s_2 D_{-1,i} \bar{D}_{-j,-j} - \lambda_m s_2 D_{-1,-j} \bar{D}_{-j,i} \\ & \quad - s_2 (s_2 D_{-1,i} \bar{\lambda}_m \bar{B}_{-i,-j} - \lambda_m s_2 D_{-1,-j} \bar{\lambda}_m \bar{B}_{-i,i}) \end{aligned}$$

Since $\bar{D}_{-j,i}, \bar{B}_{-i,-j}, \bar{B}_{-i,i} \in I_m$, it follows that $-\lambda_m s_2 D_{-1,-j} \bar{D}_{-j,i} - s_2 (s_2 D_{-1,i} \bar{\lambda}_m \bar{B}_{-i,-j} - \lambda_m s_2 D_{-1,-j} \bar{\lambda}_m \bar{B}_{-i,i}) \in I_m$. On the other hand $s_2 D_{-1,i} \bar{D}_{-j,-j} \notin I_m$ since $D_{-1,i} \notin I_m$, $\bar{D}_{-j,-j} \equiv 1 \pmod{I_m}$ and s_2 is invertible. It follows that $[\tau, g_2]_{-1,j} \notin I_m$ and hence $[\tau, g_2] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Further $[\tau, g_2]_{l1} \in I_m \forall l \in$

$\{2, \dots, n\}$ since $B \equiv 0 \pmod{I_m}$. Thus one can proceed now as in case 1.1.1 or as in case 1.1.2.

case 1.2 Assume $\sigma_{*j} \equiv e_j \pmod{I_m} \ \forall j \in \{1, \dots, n-2\}$, $\sigma_{*,n-1} \not\equiv e^{n-1} \pmod{I_m}$ and $a_{1,n-1} \in I_m$.

Consider the first row of

$$\begin{aligned}
& [\sigma, T_{1, -(n-1)}(1)] \\
& = (e + \sigma_{*1} \sigma'_{-(n-1),*} - \bar{\lambda}_m \sigma_{*,n-1} \sigma'_{-1,*}) T_{1, -(n-1)}(-1) \\
& = (e + \begin{matrix} 1 \\ n \\ -n \\ -1 \end{matrix} \begin{pmatrix} a_{11} \\ \vdots \\ a_{n-2,1} \\ 0 \\ 0 \\ c_{-n,1} \\ \vdots \\ c_{-1,1} \end{pmatrix} \begin{pmatrix} 1 & n & -n & -1 \\ \lambda_m \bar{c}_{-1,n-1} \dots \lambda_m \bar{c}_{-n,n-1} & 0 & 1 & \bar{a}_{n-2,n-1} \dots \bar{a}_{1,n-1} \end{pmatrix}) \\
& \quad - \bar{\lambda}_m \begin{matrix} 1 \\ n \\ -n \\ -1 \end{matrix} \begin{pmatrix} a_{1,n-1} \\ \vdots \\ a_{n-2,n-1} \\ 1 \\ 0 \\ c_{-n,n-1} \\ \vdots \\ c_{-1,n-1} \end{pmatrix} \begin{pmatrix} 1 & n & -n & -1 \\ \lambda_m \bar{c}_{-1,1} \dots \lambda_m \bar{c}_{-n,1} & 0 & 0 & \bar{a}_{n-2,1} \dots \bar{a}_{11} \end{pmatrix}) \\
& \quad \cdot T_{1, -(n-1)}(-1)
\end{aligned}$$

which equals

$$\begin{aligned}
& \begin{pmatrix} 1 & n & -n & -1 \\ 1 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \\
& + a_{11} \begin{pmatrix} 1 & n & -n & -1 \\ \lambda_m \bar{c}_{-1,n-1} & \dots & \lambda_m \bar{c}_{-n,n-1} & 0 & 1 & \bar{a}_{n-2,n-1} & \dots & \bar{a}_{1,n-1} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& -\bar{\lambda}_m a_{1,n-1} \begin{pmatrix} 1 & & n & & -n & & & -1 \\ \lambda_m \bar{c}_{-1,1} & \dots & \lambda_m \bar{c}_{-n,1} & 0 & 0 & \bar{a}_{n-2,1} & \dots & \bar{a}_{11} \end{pmatrix} \\
& + \begin{pmatrix} 1 & & n & & -n & & & -1 \\ 0 & \dots & 0 & 0 & x_1 & 0 & \dots & 0 & x_2 \end{pmatrix}
\end{aligned}$$

for some $x_1, x_2 \in R$. It is clearly not congruent to f_1 modulo I_m since $a_{11} \equiv 1 \pmod{I_m}$, $\sigma_{*,n-1} \not\equiv e^{n-1} \pmod{I_m}$ and $a_{1,n-1} \in I_m$. Hence $[\sigma, T_{1,-(n-1)}(1)] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Set $g_1 := T_{1,-(n-1)}(s_1) \in U_1$. By Lemma 4.5, $[\sigma, g_1] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Clearly the n -th row of $[\sigma, g_1]$ equals f_n . Let P_{1n} be as in 3.11 and set $\epsilon_2 := P_{1n} \in EU_{2n}(R_m, \Lambda_m)$. Then the first row of $\tau := {}^{\epsilon_2}[\sigma, g_1]$ equals f_1 . Since $U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ is normal, $\tau \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. One can proceed now as in case 1.1.

case 1.3 Assume $\sigma_{*j} \equiv e_j \pmod{I_m} \forall j \in \{1, \dots, n-2\}$ and $a_{1,n-1} \notin I_m$. Consider the second row of

$$\begin{aligned}
& [\sigma, T_{2,-(n-1)}(1)] \\
& = (e + \sigma_{*2} \sigma'_{-(n-1),*} - \bar{\lambda}_m \sigma_{*,n-1} \sigma'_{-2,*}) T_{1,-(n-1)}(-1) \\
& = (e + \begin{matrix} 1 \\ n \\ -n \\ -1 \end{matrix} \begin{pmatrix} a_{12} \\ \vdots \\ a_{n-2,2} \\ 0 \\ 0 \\ c_{-n,2} \\ \vdots \\ c_{-1,2} \end{pmatrix} \begin{pmatrix} 1 & & n & & -n & & & -1 \\ \lambda_m \bar{c}_{-1,n-1} & \dots & \lambda_m \bar{c}_{-n,n-1} & 0 & 1 & \bar{a}_{n-2,n-1} & \dots & \bar{a}_{1,n-1} \end{pmatrix} \\
& \quad - \bar{\lambda}_m \begin{matrix} 1 \\ n \\ -n \\ -1 \end{matrix} \begin{pmatrix} a_{1,n-1} \\ \vdots \\ a_{n-2,n-1} \\ 1 \\ 0 \\ c_{-n,n-1} \\ \vdots \\ c_{-1,n-1} \end{pmatrix} \begin{pmatrix} 1 & & n & & -n & & & -1 \\ \lambda_m \bar{c}_{-1,2} & \dots & \lambda_m \bar{c}_{-n,2} & 0 & 0 & \bar{a}_{n-2,2} & \dots & \bar{a}_{12} \end{pmatrix}) \\
& \quad \cdot T_{2,-(n-1)}(-1)
\end{aligned}$$

which equals

$$\begin{aligned}
& \begin{pmatrix} 1 & & n & -n & & & & -1 \\ 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \\
& + a_{22} \begin{pmatrix} 1 & & n & & -n & & & -1 \\ \lambda_m \bar{c}_{-1,n-1} & \dots & \lambda_m \bar{c}_{-n,n-1} & 0 & 1 & \bar{a}_{n-2,n-1} & \dots & \bar{a}_{1,n-1} \end{pmatrix} \\
& - \bar{\lambda}_m a_{2,n-1} \begin{pmatrix} 1 & & n & & -n & & & -1 \\ \lambda_m \bar{c}_{-1,2} & \dots & \lambda_m \bar{c}_{-n,2} & 0 & 0 & \bar{a}_{n-2,2} & \dots & \bar{a}_{12} \end{pmatrix} \\
& + \begin{pmatrix} 1 & & n & -n & & & & -1 \\ 0 & \dots & 0 & 0 & x_1 & 0 & \dots & 0 & x_2 & 0 \end{pmatrix}
\end{aligned}$$

for some $x_1, x_2 \in R$. Its last entry clearly does not lie in I_m . Hence $[\sigma, T_{2,-(n-1)}(1)] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Set $g_1 := T_{2,-(n-1)}(s_1) \in U_1$. By Lemma 4.5, $[\sigma, g_1] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Clearly the n -th row of $[\sigma, g_1]$ equals f_n . Set $\epsilon_2 := P_{1n} \in EU_{2n}(R_m, \Lambda_m)$. Then the first row of $\tau := {}^{\epsilon_2}[\sigma, g_1]$ equals f_1 . Since $U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ is normal, $\tau \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. One can proceed now as in case 1.1.

case 1.4 Assume $\sigma_{*j} \equiv e_j \pmod{I_m} \ \forall j \in \{1, \dots, n-1\}$, $\sigma_{*n} \not\equiv e_n \pmod{I_m}$ and $a_{1n} \in I_m$.

Consider the first row of

$$\begin{aligned}
& [\sigma, T_{1,-n}(1)] \\
& = (e + \sigma_{*1}\sigma'_{-n,*} - \bar{\lambda}_m \sigma_{*n}\sigma'_{-1,*})T_{1,-n}(-1) \\
& = (e + \begin{matrix} 1 \\ n \\ -n \\ -1 \end{matrix} \begin{pmatrix} a_{11} \\ \vdots \\ a_{n-2,1} \\ 0 \\ 0 \\ c_{-n,1} \\ \vdots \\ c_{-1,1} \end{pmatrix} \begin{pmatrix} 1 & n & -n & -1 \\ \lambda_m \bar{c}_{-1,n} & \dots & \lambda_m \bar{c}_{-n,n} & 1 & 0 & \bar{a}_{n-2,n} & \dots & \bar{a}_{1n} \end{pmatrix})
\end{aligned}$$

$$\begin{aligned}
& \begin{pmatrix} 1 & & & & \\ & a_{1n} & & & \\ & \vdots & & & \\ & a_{n-2,n} & & & \\ & 0 & & & \\ n & 1 & & & \\ & c_{-n,n} & & & \\ & \vdots & & & \\ -1 & c_{-1,n} & & & \end{pmatrix} \begin{pmatrix} 1 & n & -n & -1 \\ \lambda_m \bar{c}_{-1,1} \dots \lambda_m \bar{c}_{-n,1} & 0 & 0 & \bar{a}_{n-2,1} \dots \bar{a}_{11} \end{pmatrix} \\
& \cdot T_{1,-n}(-1)
\end{aligned}$$

which equals

$$\begin{aligned}
& \begin{pmatrix} 1 & n & -n & -1 \\ 1 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \\
& + a_{11} \begin{pmatrix} 1 & n & -n & -1 \\ \lambda_m \bar{c}_{-1,n} & \dots & \lambda_m \bar{c}_{-n,n} & 1 & 0 & \bar{a}_{n-2,n} & \dots & \bar{a}_{1n} \end{pmatrix} \\
& - \bar{\lambda}_m a_{1n} \begin{pmatrix} 1 & n & -n & -1 \\ \lambda_m \bar{c}_{-1,1} & \dots & \lambda_m \bar{c}_{-n,1} & 0 & 0 & \bar{a}_{n-2,1} & \dots & \bar{a}_{11} \end{pmatrix} \\
& + \begin{pmatrix} 1 & n & -n & -1 \\ 0 & \dots & 0 & x_1 & 0 & \dots & 0 & x_2 \end{pmatrix}
\end{aligned}$$

for some $x_1, x_2 \in R$. It is clearly not congruent to f_1 modulo I_m since $a_{11} \equiv 1 \pmod{I_m}$, $\sigma_{*n} \not\equiv e_n \pmod{I_m}$ and $a_{1n} \in I_m$. Hence $[\sigma, T_{1,-n}(1)] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Set $g_1 := T_{1,-n}(s_1) \in U_1$. By Lemma 4.5, $[\sigma, g_1] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Clearly the $(n-1)$ -th row of $[\sigma, g_1]$ equals f_{n-1} . Set $\epsilon_2 := P_{1(n-1)} \in EU_{2n}(R_m, \Lambda_m)$. Then the first row of $\tau := {}^{\epsilon_2}[\sigma, g_1]$ equals f_1 . Since $U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ is normal, $\tau \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. One can proceed now as in case 1.1.

case 1.5 Assume $\sigma_{*j} \equiv e_j \pmod{I_m} \ \forall j \in \{1, \dots, n-1\}$ and $a_{1n} \notin I_m$. Consider the second row of

$$\begin{aligned}
& [\sigma, T_{2,-n}(1)] \\
& = (e + \sigma_{*2}\sigma'_{-n,*} - \bar{\lambda}_m \sigma_{*n}\sigma'_{-2,*})T_{2,-n}(-1)
\end{aligned}$$

$$\begin{pmatrix} 1 \\ \vdots \\ a_{n-2,n} \\ 0 \\ 1 \\ \vdots \\ c_{-n,n} \\ \vdots \\ c_{-1,n} \end{pmatrix} \begin{pmatrix} 1 & n & -n & -1 \\ \lambda_m \bar{c}_{-1,2} \dots \lambda_m \bar{c}_{-n,2} & 0 & 0 & \bar{a}_{n-2,2} \dots \bar{a}_{12} \end{pmatrix}$$

which equals

$$\begin{aligned}
& \begin{pmatrix} 1 & & n & -n & & & -1 \\ 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \\
& + a_{22} \begin{pmatrix} 1 & & n & & -n & & & -1 \\ \lambda_m \bar{c}_{-1,n} & \dots & \lambda_m \bar{c}_{-n,n} & 1 & 0 & \bar{a}_{n-2,n} & \dots & \bar{a}_{1n} \end{pmatrix} \\
& - \bar{\lambda}_m a_{2n} \begin{pmatrix} 1 & & n & & -n & & & -1 \\ \lambda_m \bar{c}_{-1,2} & \dots & \lambda_m \bar{c}_{-n,2} & 0 & 0 & \bar{a}_{n-2,2} & \dots & \bar{a}_{12} \end{pmatrix} \\
& + \begin{pmatrix} 1 & & n & -n & & & & -1 \\ 0 & \dots & 0 & x_1 & 0 & \dots & 0 & x_2 & 0 \end{pmatrix}
\end{aligned}$$

for some $x_1, x_2 \in R$. Its last entry does clearly not lie in I_m and hence $[\sigma, T_{2,-n}(1)] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Set $g_1 := T_{2,-n}(s_1) \in U_1$. By Lemma 4.5, $[\sigma, g_1] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Clearly the $(n-1)$ -th row of $[\sigma, g_1]$ equals f_{n-1} . Set $\epsilon_2 := P_{1(n-1)} \in EU_{2n}(R_m, \Lambda_m)$. Then the first row of $\tau := \epsilon_2[\sigma, g_1]$ equals f_1 . Since $U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ is normal, $\tau \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. One can proceed now as in case 1.1.

case 2 Assume that $\alpha, \delta \equiv e^{n \times n} \pmod{I_m}$ and $\gamma \equiv 0 \pmod{I_m}$.

Recall that $\sigma = {}^{\epsilon_1}[\epsilon_0 h, g_0] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ has the form

$$\left(\begin{array}{cc|cc} \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \\ 0 & e^{2 \times 2} & \beta_3 & \beta_4 \\ \hline & \gamma & & \delta \end{array} \right) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where $\alpha_1, \beta_2 \in M_{n-2}(R_m)$, $\alpha_2, \beta_1 \in M_{(n-2) \times 2}(R_m)$, $\beta_3 \in M_2(R_m)$, $\beta_4 \in M_{2 \times (n-2)}(R_m)$ and $\alpha, \beta, \gamma, \delta \in M_n(R_m)$. Clearly $\beta \not\equiv 0 \pmod{I_m}$ since $\sigma \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$.

case 2.1 Assume that $\beta_3 \not\equiv 0 \pmod{I_m}$ or $\beta_4 \not\equiv 0 \pmod{I_m}$.

Set $g_1 := T_{-n, n-1}(s_1) \in U_1$ and $\omega := [\sigma^{-1}, g_1]$. Then

$$\begin{aligned} & \omega \\ &= [\sigma^{-1}, g_1] \\ &= (e + \sigma'_{*, -n} s_1 \sigma_{(n-1)*} - \sigma'_{*, -(n-1)} \lambda_m s_1 \sigma_{n*})(g_1)^{-1} \\ &= (e + s_1 \begin{pmatrix} 1 & \bar{\lambda} \bar{b}_{n, -1} \\ & \vdots \\ n & \bar{\lambda} \bar{b}_{n, -n} \\ -n & 1 \\ & 0 \\ & \vdots \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & n & -n & & -1 \\ 0 & \dots & 0 & 1 & 0 & b_{n-1, -n} & \dots & b_{n-1, -1} \end{pmatrix} \\ & \quad - \lambda_m s_1 \begin{pmatrix} 1 & \bar{\lambda} \bar{b}_{n-1, -1} \\ & \vdots \\ n & \bar{\lambda} \bar{b}_{n-1, -n} \\ -n & 0 \\ & 1 \\ & 0 \\ & \vdots \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & n & -n & & -1 \\ 0 & \dots & 0 & 1 & b_{n, -n} & \dots & b_{n, -1} \end{pmatrix}) \\ & \quad \cdot (g_1)^{-1}. \end{aligned}$$

Since $\beta_3 \not\equiv 0 \pmod{I_m}$ or $\beta_4 \not\equiv 0 \pmod{I_m}$, $(\omega_{-n, -n}, \dots, \omega_{-n, -1}) \not\equiv (1, 0, \dots, 0) \pmod{I_m}$ or $(\omega_{-(n-1), -n}, \dots, \omega_{-(n-1), -1}) \not\equiv (0, 1, 0, \dots, 0) \pmod{I_m}$. Hence $\omega \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Further the next to last row of ω equals f_{-2} . Set $\epsilon_2 := P_{1, -2} \in EU_{2n}(R_m, \Lambda_m)$. Then the first row of ${}^{\epsilon_2}\omega$ equals f_1 . Since $U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ is normal, ${}^{\epsilon_2}\omega \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. One can proceed now as in case 1.1 (${}^{\epsilon_2}\omega$ has the same properties as τ in case 1.1).

case 2.2 Assume that $\beta_3 \equiv 0(\text{mod } I_m)$ and $\beta_4 \equiv 0(\text{mod } I_m)$.

It follows that $\beta_1 \equiv 0(\text{mod } I_m)$. Hence $\beta_2 \not\equiv 0(\text{mod } I_m)$ since $\beta \not\equiv 0(\text{mod } I_m)$. Set

$$\xi := \prod_{k=-1}^{-(n-2)} T_{n-1,k}(-b_{n-1,k}) \prod_{k=-1}^{-(n-2)} T_{nk}(-b_{nk}) \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m)).$$

Then $\omega := \sigma\xi$ has the form

$$\left(\begin{array}{cc|cc} \alpha'_1 & \alpha'_2 & \beta'_1 & \beta'_2 \\ 0 & e^{2 \times 2} & \beta'_3 & 0 \\ \hline & \gamma' & & \delta' \end{array} \right) = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$$

where $\alpha'_1, \beta'_2 \in M_{n-2}(R_m)$, $\alpha'_2, \beta'_1 \in M_{(n-2) \times 2}(R_m)$, $\beta'_3 \in M_2(R_m)$ and $\alpha', \beta', \gamma', \delta' \in M_n(R_m)$. Further $\alpha', \delta' \equiv e^{n \times n}(\text{mod } I_m)$, $\gamma' \equiv 0(\text{mod } I_m)$, $\beta'_1 \equiv 0(\text{mod } I_m)$, $\beta'_3 \equiv 0(\text{mod } I_m)$ and $\beta'_2 \not\equiv 0(\text{mod } I_m)$. Since $\beta'_2 \not\equiv 0(\text{mod } I_m)$, there is an $i \in \{1, \dots, n-2\}$ and a $j \in \{-(n-2), \dots, -1\}$ such that $\beta'_{ij} \notin I_m$. Choose an $l \in \{1, \dots, n-2\} \setminus \{-j\}$ and set $\epsilon_{11} := T_{jl}(-1) \in EU_{2n}(R_m, \Lambda_m)$. Then $\epsilon_{11}\omega$ has the form

$$\left(\begin{array}{cc|cc} \alpha''_1 & \alpha''_2 & \beta''_1 & \beta''_2 \\ 0 & e^{2 \times 2} & \beta''_3 & \beta''_4 \\ \hline & \gamma'' & & \delta'' \end{array} \right) = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix}$$

where $\alpha''_1, \beta''_2 \in M_{n-2}(R_m)$, $\alpha''_2, \beta''_1 \in M_{(n-2) \times 2}(R_m)$, $\beta''_3 \in M_2(R_m)$, $\beta''_4 \in M_{2 \times (n-2)}(R_m)$ and $\alpha'', \beta'', \gamma'', \delta'' \in M_n(R_m)$. Further $\alpha''_{il} \not\equiv \delta_{il}(\text{mod } I_m)$. Hence $\alpha'' \not\equiv e^{n \times n}(\text{mod } I_m)$ and thus one can proceed as in case 1.1.

Part II Assume that $h \notin CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ and $n = 3$.

There is a $g_0 \in U_0$ and $\epsilon_0, \epsilon_1 \in EU_6(R_m, \Lambda_m)$ such that $\sigma := \epsilon_1[\epsilon_0 h, g_0] \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ and σ has the form

$$\left(\begin{array}{ccc|c} * & & & \beta \\ 0 & 0 & 1 & \\ \hline & \gamma & & \delta \end{array} \right) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where $\alpha = (\alpha_{ij})_{1 \leq i, j \leq 3}$, $\beta = (\beta_{ij})_{\substack{1 \leq i \leq 3 \\ -3 \leq j \leq -1}}$, $\gamma = (\gamma_{ij})_{\substack{-3 \leq i \leq -1 \\ 1 \leq j \leq 3}}$, $\delta = (\delta_{ij})_{-3 \leq i, j \leq -1} \in M_3(R_m)$ (see Part I above and [2], chapter IV, Lemma 3.12, Part II, general case).

case 1 Assume that there is an $i \in \{-3, -2, -1\}$ such that $\gamma_{i2} \notin I_m$.

Set $g_1 := T_{1,-2}(s_1) \in U_1$ and $\omega := [\sigma, g_1]$. Then

$$\begin{aligned}
& \omega \\
&= [\sigma, g_1] \\
&= (e + \sigma_{*1} s_1 \sigma'_{-2,*} - \sigma_{*2} \bar{\lambda}_m s_1 \sigma'_{-1,*}) (g_1)^{-1} \\
&= (e + s_1 \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ 0 \\ \gamma_{-3,1} \\ \gamma_{-2,1} \\ \gamma_{-1,1} \end{pmatrix} \begin{pmatrix} \lambda_m \bar{\gamma}_{-1,2} & \lambda_m \bar{\gamma}_{-2,2} & \lambda_m \bar{\gamma}_{-3,2} & 0 & \bar{\alpha}_{22} & \bar{\alpha}_{12} \end{pmatrix} \\
&\quad - \bar{\lambda}_m s_1 \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \\ 0 \\ \gamma_{-3,2} \\ \gamma_{-2,2} \\ \gamma_{-1,2} \end{pmatrix} \begin{pmatrix} \lambda_m \bar{\gamma}_{-1,1} & \lambda_m \bar{\gamma}_{-2,1} & \lambda_m \bar{\gamma}_{-3,1} & 0 & \bar{\alpha}_{21} & \bar{\alpha}_{11} \end{pmatrix}) \\
&\quad \cdot (g_1)^{-1}.
\end{aligned}$$

Assume that

$$\begin{aligned}
& s_1 \sigma_{*1} \begin{pmatrix} \lambda_m \bar{\gamma}_{-1,2} & \lambda_m \bar{\gamma}_{-2,2} & \lambda_m \bar{\gamma}_{-3,2} \end{pmatrix} \\
& - \bar{\lambda}_m s_1 \sigma_{*2} \begin{pmatrix} \lambda_m \bar{\gamma}_{-1,1} & \lambda_m \bar{\gamma}_{-2,1} & \lambda_m \bar{\gamma}_{-3,1} \end{pmatrix} \equiv 0 \pmod{I_m}.
\end{aligned}$$

By multiplying σ'_{1*} from the left we get that $s_1 \begin{pmatrix} \lambda_m \bar{\gamma}_{-1,2} & \lambda_m \bar{\gamma}_{-2,2} & \lambda_m \bar{\gamma}_{-3,2} \end{pmatrix} \equiv 0 \pmod{I_m}$ which implies $\begin{pmatrix} \gamma_{-1,2} & \gamma_{-2,2} & \gamma_{-3,2} \end{pmatrix} \equiv 0 \pmod{I_m}$. Since that is a contradiction,

$$\begin{aligned}
& s_1 \sigma_{*1} \begin{pmatrix} \lambda_m \bar{\gamma}_{-1,2} & \lambda_m \bar{\gamma}_{-2,2} & \lambda_m \bar{\gamma}_{-3,2} \end{pmatrix} \\
& - \bar{\lambda}_m s_1 \sigma_{*2} \begin{pmatrix} \lambda_m \bar{\gamma}_{-1,1} & \lambda_m \bar{\gamma}_{-2,1} & \lambda_m \bar{\gamma}_{-3,1} \end{pmatrix} \not\equiv 0 \pmod{I_m}
\end{aligned}$$

and hence $\omega \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Further the third row of ω equals f_3 . Set $\epsilon_2 := P_{13} \in EU_6(R_m, \Lambda_m)$. Then the first row of ${}^{\epsilon_2}\omega$ equals f_1 . Since $U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ is normal, ${}^{\epsilon_2}\omega \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. One can proceed now as in Part I, case 1 (${}^{\epsilon_2}\omega$ has the same properties as τ in Part I, case 1).

case 2 Assume that there is an $i \in \{-3, -2, -1\}$ such that $\gamma_{i1} \notin I_m$.

This case can be treated similarly.

case 3 Assume that $\gamma_{-3,1}, \gamma_{-3,2}, \gamma_{-2,1}, \gamma_{-2,2}, \gamma_{-1,1}, \gamma_{-1,2} \in I_m$ and one of the entries $\beta_{3,-3}$ and $\beta_{3,-2}$ does not lie in I_m .

By [2], chapter IV, Lemma 3.12, Part II, general case there are $x_1, x_2 \in I_m$ such that $\gamma_{-1,1} + x_2(x_1 \gamma_{-1,1} + \gamma_{-2,1}) \in \text{rad}(R_m) \cap I_m$ where $\text{rad}(R_m)$ is the Jacobson radical

of the ring R_m . Set $\xi_1 := T_{-1,-2}(x_2)T_{-2,-1}(x_1) \in EU_6((R_m, \Lambda_m), (I_m, \Gamma_m))$. Then $\rho := \xi_1\sigma$ has the form

$$\left(\begin{array}{ccc|c} * & & & \beta' \\ 0 & 0 & 1 & \\ \hline & \gamma' & & \delta' \end{array} \right) = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$$

where $\alpha' = (\alpha'_{ij})_{1 \leq i, j \leq 3}$, $\beta' = (\beta'_{ij})_{\substack{1 \leq i \leq 3 \\ -3 \leq j \leq -1}}$, $\gamma' = (\gamma'_{ij})_{\substack{-3 \leq i \leq -1 \\ 1 \leq j \leq 3}}$, $\delta' = (\delta'_{ij})_{-3 \leq i, j \leq -1} \in M_3(R_m)$. Further $(\beta'_{3,-3} \notin I_m \vee \beta'_{3,-2} \notin I_m) \wedge \gamma'_{-1,1} \in \text{rad}(R_m) \cap I_m$. Set $g_1 := T_{13}(s_1) \in U_1$ and $\omega := [\rho^{-1}, g_1]$. Then

$$\begin{aligned} & \omega \\ &= [\rho^{-1}, g_1] \\ &= (e + \rho'_{*1}s_1\rho_{3*} - \rho'_{*,-3}s_1\rho_{-1,*})(g_1)^{-1} \\ &= (e + s_1 \begin{pmatrix} \bar{\delta}'_{-1,-1} \\ \bar{\delta}'_{-1,-2} \\ \bar{\delta}'_{-1,-3} \\ \lambda\bar{\gamma}'_{-1,3} \\ \lambda\bar{\gamma}'_{-1,2} \\ \lambda\bar{\gamma}'_{-1,1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & \beta'_{3,-3} & \beta'_{3,-2} & \beta'_{3,-1} \end{pmatrix} \\ & \quad - s_1 \begin{pmatrix} \bar{\lambda}\bar{\beta}_{3,-1} \\ \bar{\lambda}\bar{\beta}_{3,-2} \\ \bar{\lambda}\bar{\beta}_{3,-3} \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \gamma'_{-1,1} & \gamma'_{-1,2} & \gamma'_{-1,3} & \delta'_{-1,-3} & \delta'_{-1,-2} & \delta'_{-1,-1} \end{pmatrix}) \\ & \quad \cdot (g_1)^{-1}. \end{aligned}$$

Assume that

$$\begin{aligned} & s_1\rho'_{*1} \begin{pmatrix} \beta'_{3,-3} & \beta'_{3,-2} \end{pmatrix} \\ & - s_1\rho'_{*,-3} \begin{pmatrix} \delta'_{-1,-3} & \delta'_{-1,-2} \end{pmatrix} \equiv 0 \pmod{I_m}. \end{aligned}$$

By multiplying ρ_{1*} from the left we get that $s_1 \begin{pmatrix} \beta'_{3,-3} & \beta'_{3,-2} \end{pmatrix} \equiv 0 \pmod{I_m}$ which implies $\begin{pmatrix} \beta'_{3,-3} & \beta'_{3,-2} \end{pmatrix} \equiv 0 \pmod{I_m}$. Since that is a contradiction,

$$\begin{aligned} & s_1\rho'_{*1} \begin{pmatrix} \beta'_{3,-3} & \beta'_{3,-2} \end{pmatrix} \\ & - s_1\rho'_{*,-3} \begin{pmatrix} \delta'_{-1,-3} & \delta'_{-1,-2} \end{pmatrix} \not\equiv 0 \pmod{I_m} \end{aligned}$$

and hence $\omega \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Obviously $\omega_{-1,*} \equiv f_{-1} \pmod{I_m}$ and $\omega_{-1,-1} \equiv 1 \pmod{\text{rad}(R_m) \cap I_m}$. Set $\epsilon_2 := P_{3,-1} \in EU_6(R_m, \Lambda_m)$ and $\zeta := \epsilon_2\omega$. Then $\zeta \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Further $\zeta_{3*} \equiv f_3 \pmod{I_m}$ and $\zeta_{33} \equiv 1 \pmod{\text{rad}(R_m) \cap I_m}$. By Nakayama's lemma ζ_{33} is invertible. Set $\xi_2 := T_{32}(-(\zeta_{33})^{-1}\zeta_{32})T_{31}(-(\zeta_{33})^{-1}\zeta_{31})T_{3,-1}(-(\zeta_{33})^{-1}\zeta_{3,-1})T_{3,-2}(-(\zeta_{33})^{-1}\zeta_{3,-2}) \in EU_6((R_m, \Lambda_m), (I_m, \Gamma_m))$ and $\eta := \zeta\xi_2$. Then η has the form

$$\left(\begin{array}{ccc|ccc} * & & & * & & \\ 0 & 0 & \alpha''_{33} & \beta''_{3,-3} & 0 & 0 \\ \hline & \gamma'' & & & \delta'' & \end{array} \right) = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix}$$

where $\alpha'' = (\alpha''_{ij})_{1 \leq i, j \leq 3}$, $\beta'' = (\beta''_{ij})_{\substack{1 \leq i \leq 3 \\ -3 \leq j \leq -1}}$, $\gamma'' = (\gamma''_{ij})_{\substack{-3 \leq i \leq -1 \\ 1 \leq j \leq 3}}$, $\delta'' = (\delta''_{ij})_{-3 \leq i, j \leq -1} \in M_3(R_m)$. Since $\zeta \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, $\eta \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Further $\alpha''_{33} \equiv 1 \pmod{\text{rad}(R_m) \cap I_m}$ and $\beta''_{3,-3} \in I_m$. Since $\eta_{3*} \equiv f_3 \pmod{I_m}$, $\eta_{*, -3} \equiv e_{-3} \pmod{I_m}$ (apply Lemma 3.9 to the image of η in $U_{2n}(R_m/I_m, \Lambda_m/(\Lambda_m \cap I_m))$). Hence $\beta''_{1,-3}, \beta''_{2,-3}, \delta''_{-3,-3} - 1, \delta''_{-2,-3}, \delta''_{-1,-3} \in I_m$.

case 3.1 Assume that there is an $i \in \{-3, -2, -1\}$ and a $j \in \{1, 2\}$ such that $\gamma''_{ij} \notin I_m$. See case 1.

case 3.2 Assume that $\gamma''_{-3,1}, \gamma''_{-3,2}, \gamma''_{-2,1}, \gamma''_{-2,2}, \gamma''_{-1,1}, \gamma''_{-1,2} \in I_m$ and one of the entries $\beta''_{1,-2}, \beta''_{1,-1}, \beta''_{2,-2}, \beta''_{2,-1}, \delta''_{-3,-2}$ and $\delta''_{-3,-1}$ does not lie in I_m .

Set $g_2 := T_{-1,2}(s_2) \in U_2$ and $\theta := [\eta, g_2]$. Then

$$\begin{aligned} & \theta \\ &= [\eta, g_2] \\ &= (e + \eta_{*, -1} s_2 \eta'_{2*} - \eta_{*, -2} \lambda_m s_2 \eta'_{1*})(g_2)^{-1} \\ &= (e + s_2 \begin{pmatrix} \beta''_{1,-1} \\ \beta''_{2,-1} \\ 0 \\ \delta''_{-3,-1} \\ \delta''_{-2,-1} \\ \delta''_{-1,-1} \end{pmatrix} \begin{pmatrix} \bar{\delta}''_{-1,-2} & \bar{\delta}''_{-2,-2} & \bar{\delta}''_{-3,-2} & 0 & \bar{\lambda}_m \bar{\beta}''_{2,-2} & \bar{\lambda}_m \bar{\beta}''_{1,-2} \end{pmatrix} \\ & \quad - \lambda_m s_2 \begin{pmatrix} \beta''_{1,-2} \\ \beta''_{2,-2} \\ 0 \\ \delta''_{-3,-2} \\ \delta''_{-2,-2} \\ \delta''_{-1,-2} \end{pmatrix} \begin{pmatrix} \bar{\delta}''_{-1,-1} & \bar{\delta}''_{-2,-1} & \bar{\delta}''_{-3,-1} & 0 & \bar{\lambda}_m \bar{\beta}''_{2,-1} & \bar{\lambda}_m \bar{\beta}''_{1,-1} \end{pmatrix}) \\ & \quad \cdot (g_2)^{-1}. \end{aligned}$$

Assume that

$$\begin{aligned} & s_2 \eta_{*, -1} \begin{pmatrix} \bar{\delta}''_{-3,-2} & 0 & \bar{\lambda}_m \bar{\beta}''_{2,-2} & \bar{\lambda}_m \bar{\beta}''_{1,-2} \end{pmatrix} \\ & - \lambda_m s_2 \eta_{*, -2} \begin{pmatrix} \bar{\delta}''_{-3,-1} & 0 & \bar{\lambda}_m \bar{\beta}''_{2,-1} & \bar{\lambda}_m \bar{\beta}''_{1,-1} \end{pmatrix} \equiv 0 \pmod{I_m}. \end{aligned}$$

It follows that $(\delta''_{-3,-2} \ 0 \ \beta''_{2,-2} \ \beta''_{1,-2}), (\delta''_{-3,-1} \ 0 \ \beta''_{2,-1} \ \beta''_{1,-1}) \equiv 0 \pmod{I_m}$. Since that is a contradiction,

$$\begin{aligned} & s_2 \eta_{*, -1} \begin{pmatrix} \bar{\delta}''_{-3,-2} & 0 & \bar{\lambda}_m \bar{\beta}''_{2,-2} & \bar{\lambda}_m \bar{\beta}''_{1,-2} \end{pmatrix} \\ & - \lambda_m s_2 \eta_{*, -2} \begin{pmatrix} \bar{\delta}''_{-3,-1} & 0 & \bar{\lambda}_m \bar{\beta}''_{2,-1} & \bar{\lambda}_m \bar{\beta}''_{1,-1} \end{pmatrix} \not\equiv 0 \pmod{I_m} \end{aligned}$$

and hence $\theta \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Clearly $\theta_{3*} = f_3$. Set $\epsilon_3 := P_{13}$. Thus the first row of ${}^{\epsilon_3}\theta$ equals f_1 . Since ${}^{\epsilon_3}\theta \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, one can proceed now as in Part I, case 1.

case 3.3 Assume that $\gamma''_{-3,1}, \gamma''_{-3,2}, \gamma''_{-2,1}, \gamma''_{-2,2}, \gamma''_{-1,1}, \gamma''_{-1,2}, \beta''_{1,-2}, \beta''_{1,-1}, \beta''_{2,-2}, \beta''_{2,-1}, \delta''_{-3,-2}, \delta''_{-3,-1} \in I_m$ and one of the elements $\alpha''_{11} - 1, \alpha''_{21}, \gamma''_{-1,3}, \delta''_{-1,-1} - 1$ and $\delta''_{-1,-2}$ does not lie in I_m .

Set $g_2 := T_{13}(s_2) \in U_2$ and $\theta := [\eta^{-1}, g_2]$. Then

$$\begin{aligned}
& \theta \\
&= [\eta^{-1}, g_2] \\
&= (e + \eta'_{*1} s_2 \eta_{3*} - \eta'_{*,-3} s_2 \eta_{-1,*})(g_2)^{-1} \\
&= (e + s_2 \begin{pmatrix} \bar{\delta}''_{-1,-1} \\ \bar{\delta}''_{-1,-2} \\ \bar{\delta}''_{-1,-3} \\ \lambda_m \gamma''_{-1,3} \\ \lambda_m \gamma''_{-1,2} \\ \lambda_m \gamma''_{-1,1} \end{pmatrix} \begin{pmatrix} 0 & 0 & \alpha''_{33} & \beta''_{3,-3} & 0 & 0 \end{pmatrix} \\
&\quad - s_2 \begin{pmatrix} 0 \\ 0 \\ \bar{\lambda}_m \bar{\beta}''_{3,-3} \\ \bar{\alpha}''_{33} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \gamma''_{-1,1} & \gamma''_{-1,2} & \gamma''_{-1,3} & \delta''_{-1,-3} & \delta''_{-1,-2} & \delta''_{-1,-1} \end{pmatrix}) \\
&\quad \cdot (g_2)^{-1}.
\end{aligned}$$

Assume that $\theta \in U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Then $s_2 \bar{\delta}''_{-1,-1} \alpha''_{33} - s_2 = \theta_{13} \in I_m$ and $s_2 \bar{\delta}''_{-1,-2} \alpha''_{33} = \theta_{23} \in I_m$. Since $\alpha''_{33} \equiv 1 \pmod{I_m}$, it follows that $\delta''_{-1,-1} - 1, \delta''_{-1,-2} \in I_m$. Consider the column

$$\eta'_{*1} s_2 \alpha''_{33} - \eta'_{*,-3} s_2 \gamma''_{-1,3} - \begin{pmatrix} s_2 & 0 & -s_2^2 \bar{\lambda}_m \bar{\beta}''_{3,-3} \gamma''_{-1,1} & -s_2^2 \bar{\alpha}''_{33} \gamma''_{-1,1} & 0 & 0 \end{pmatrix}^t.$$

Since by assumption $\theta \in U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, the column above is congruent to 0 modulo I_m . By multiplying $\eta_{-3,*}$ from the left we get that $\gamma''_{-1,3} \in I_m$ since $\gamma''_{-3,1}, \gamma''_{-1,1} \in I_m$ ($\gamma''_{-3,1}$ is the first entry of $\eta_{-3,*}$). Hence $\eta_{-1,*} \equiv f_{-1} \pmod{I_m}$. It follows that $\eta_{*1} \equiv e_1 \pmod{I_m}$ (apply Lemma 3.9 to the image of η in $U_{2n}(R_m/I_m, \Lambda_m/(\Lambda_m \cap I_m))$) and hence $\alpha''_{11} - 1, \alpha''_{21} \in I_m$. Since that is a contradiction, $\theta \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$.

case 3.3.1 Assume that $\theta_{13} \notin I_m$ or $\theta_{23} \notin I_m$.

Set $g_3 := T_{-2,1}(s_3) \in U_3$ and $\tau := [\theta^{-1}, g_3]$. Then

$$\begin{aligned}
& \tau \\
&= [\theta^{-1}, g_3] \\
&= (e + \theta'_{*,-2} s_3 \theta_{1*} - \theta'_{*,-1} \lambda_m s_3 \theta_{2*})(g_3)^{-1}
\end{aligned}$$

$$\begin{aligned}
&= (e + s_3 \begin{pmatrix} \bar{\lambda}_m \bar{\theta}_{2,-1} \\ 0 \\ \bar{\lambda}_m \bar{\theta}_{2,-3} \\ \bar{\theta}_{23} \\ \bar{\theta}_{22} \\ \bar{\theta}_{21} \end{pmatrix} \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{1,-3} & 0 & \theta_{1,-1} \end{pmatrix} \\
&\quad - \lambda_m s_3 \begin{pmatrix} \bar{\lambda}_m \bar{\theta}_{1,-1} \\ 0 \\ \bar{\lambda}_m \bar{\theta}_{1,-3} \\ \bar{\theta}_{13} \\ \bar{\theta}_{12} \\ \bar{\theta}_{11} \end{pmatrix} \begin{pmatrix} \theta_{21} & \theta_{22} & \theta_{23} & \theta_{2,-3} & 0 & \theta_{2,-1} \end{pmatrix}) \\
&\quad \cdot (g_3)^{-1}.
\end{aligned}$$

Assume that $\tau \in U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Then $\theta'_{*, -2} s_3 \theta_{13} - \theta'_{*, -1} \lambda_m s_3 \theta_{23} \equiv 0 \pmod{I_m}$. It follows that $\theta_{13}, \theta_{23} \in I_m$ which is a contradiction. Hence $\tau \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Clearly $\tau_{2*} = f_2$. Set $\epsilon_4 := P_{12}$. Then the first row of $\epsilon_4 \theta$ equals f_1 . Since $\epsilon_4 \theta \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, one can proceed now as in Part I, case 1.

case 3.3.2 Assume that $\theta_{13} \in I_m$ and $\theta_{23} \in I_m$.

Let $\hat{\theta}$ be the image of θ in $U_{2n}(R_m/I_m, \Lambda_m/(\Lambda_m \cap I_m))$. Clearly $\hat{\theta}$ has the form

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{\theta}_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & \hat{\theta}_{-3,3} & \hat{\theta}_{-3,-3} & \hat{\theta}_{-3,-2} & \hat{\theta}_{-3,-1} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

It follows that $\hat{\theta}_{33}$ is invertible. Let \mathfrak{h} be the map defined in Definition 3.3. Then

$$\begin{aligned}
&\bar{\theta}_{33} \hat{\theta}_{-3,-2} \\
&= \mathfrak{h}(\hat{\theta}_{*3}, \hat{\theta}_{*, -2}) \\
&= \mathfrak{h}(\hat{\theta}_{e_3}, \hat{\theta}_{e_{-2}}) \\
&= \mathfrak{h}(e_3, e_{-2}) \\
&= 0.
\end{aligned}$$

Hence $\hat{\theta}_{-3,-2} = 0$ and therefore $\theta_{-3,-2} \in I_m$. Further

$$\begin{aligned}
& \bar{\theta}_{33}\hat{\theta}_{-3,-1} \\
&= \mathbb{h}(\hat{\theta}_{*3}, \hat{\theta}_{*, -1}) \\
&= \mathbb{h}(\hat{\theta}e_3, \hat{\theta}e_{-1}) \\
&= \mathbb{h}(e_3, e_{-1}) \\
&= 0.
\end{aligned}$$

Hence $\hat{\theta}_{-3,-1} = 0$ and therefore $\theta_{-3,-1} \in I_m$. Clearly $\theta_{22} = 1$. Set $\xi_3 := T_{23}(-\theta_{23})$. $T_{2,-1}(-\theta_{2,-1}) \in U_6((R_m, \Lambda_m), (I_m, \Gamma_m))$ and $\chi := \theta\xi_3$. Then $\chi_{23}, \chi_{2,-1} = 0$ and the image $\hat{\chi}$ of χ in $U_6(R_m/I_m, \Lambda_m/(\Lambda_m \cap I_m))$ has the form

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{\chi}_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & \hat{\chi}_{-3,3} & \hat{\chi}_{-3,-3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

Since $\theta \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, $\chi \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ (i.e. $\hat{\chi} \neq e$). Set $g_3 := T_{31}(s_3) \in U_3$ and $\mu := [\chi, g_3]$. Then

$$\begin{aligned}
& \mu \\
&= [\chi, g_3] \\
&= (e + \chi_{*3}s_3\chi'_{1*} - \chi_{*, -1}s_3\chi'_{-3,*})(g_3)^{-1} \\
&= (e + s_3 \begin{pmatrix} \chi_{13} \\ 0 \\ \chi_{33} \\ \chi_{-3,3} \\ \chi_{-2,3} \\ \chi_{-1,3} \end{pmatrix} \begin{pmatrix} \bar{\chi}_{-1,-1} & \bar{\chi}_{-2,-1} & \bar{\chi}_{-3,-1} & \bar{\lambda}_m\bar{\chi}_{3,-1} & 0 & \bar{\lambda}_m\bar{\chi}_{1,-1} \end{pmatrix} \\
&\quad - s_3 \begin{pmatrix} \chi_{1,-1} \\ 0 \\ \chi_{3,-1} \\ \chi_{-3,-1} \\ \chi_{-2,-1} \\ \chi_{-1,-1} \end{pmatrix} \begin{pmatrix} \lambda_m\bar{\chi}_{-1,3} & \lambda_m\bar{\chi}_{-2,3} & \lambda_m\bar{\chi}_{-3,3} & \bar{\chi}_{33} & 0 & \bar{\chi}_{13} \end{pmatrix}) \\
&\quad \cdot (g_3)^{-1}.
\end{aligned}$$

Assume that $\mu \in U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Then $s_3\chi_{33}\bar{\chi}_{-1,-1} - s_3\chi_{3,-1}\lambda_m\bar{\chi}_{-1,3} - s_3(1 + s_3\chi_{33}\bar{\chi}_{-3,-1} - s_3\chi_{3,-1}\lambda_m\bar{\chi}_{-3,3}) = \mu_{31} \in I_m$ and hence $s_3\chi_{33}\bar{\chi}_{-1,-1} - s_3 \in I_m$. It follows that $\chi_{33} \equiv 1 \pmod{I_m}$ (i.e. $\hat{\chi}_{33} = 1$) since $s_3 \in (R_m)^*$ and $\chi_{-1,-1} \equiv$

$1(\text{mod } I_m)$. That implies

$$\begin{aligned}
& \hat{\chi}_{-3,-3} \\
&= \mathbb{h}(\hat{\chi}_{*3}, \hat{\chi}_{*, -3}) \\
&= \mathbb{h}(\hat{\chi}e_3, \hat{\chi}e_{-3}) \\
&= \mathbb{h}(e_3, e_{-3}) \\
&= 1.
\end{aligned}$$

Further $s_3\chi_{-3,3}\bar{\chi}_{-1,-1} - s_3\chi_{-3,-1}\lambda_m\bar{\chi}_{-1,3} - s_3(s_3\chi_{-3,3}\bar{\chi}_{-3,-1} - s_3\chi_{-3,-1}\lambda_m\bar{\chi}_{-3,3}) = \mu_{-3,1} \in I_m$ and hence $s_3\chi_{-3,3}\bar{\chi}_{-1,-1} \in I_m$. It follows that $\chi_{-3,3} \equiv 0(\text{mod } I_m)$ (i.e. $\hat{\chi}_{-3,3} = 0$) since $s_3 \in (R_m)^*$ and $\chi_{-1,-1} \equiv 1(\text{mod } I_m)$. But that implies the contradiction $\hat{\chi} = e$. Hence $\mu \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Clearly $\mu_{2*} = f_2$. Set $\epsilon_4 := P_{12}$. Then the first row of ${}^{\epsilon_4}\mu$ equals f_1 . Since ${}^{\epsilon_4}\mu \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, one can proceed as in Part I, case 1.

case 3.4 Assume that $\gamma''_{-3,1}, \gamma''_{-3,2}, \gamma''_{-2,1}, \gamma''_{-2,2}, \gamma''_{-1,1}, \gamma''_{-1,2}, \beta''_{1,-2}, \beta''_{1,-1}, \beta''_{2,-2}, \beta''_{2,-1}, \delta''_{-3,-2}, \delta''_{-3,-1}, \alpha''_{11} - 1, \alpha''_{21}, \gamma''_{-1,3}, \delta''_{-1,-1} - 1, \delta''_{-1,-2} \in I_m$ and one of the elements $\alpha''_{12}, \alpha''_{22} - 1, \gamma''_{-2,3}, \delta''_{-2,-1}, \delta''_{-2,-2} - 1, \alpha''_{13}$ and α''_{23} does not lie in I_m . Set $g_2 := T_{21}(s_2) \in U_2$ and $\theta := [\eta, g_2]$. Then

$$\begin{aligned}
& \theta \\
&= [\eta, g_2] \\
&= (e + \eta_{*2}s_2\eta'_{1*} - \eta_{*, -1}s_2\eta'_{-2,*})(g_2)^{-1} \\
&= (e + s_2 \begin{pmatrix} \alpha''_{12} \\ \alpha''_{22} \\ 0 \\ \gamma''_{-3,2} \\ \gamma''_{-2,2} \\ \gamma''_{-1,2} \end{pmatrix} \begin{pmatrix} \bar{\delta}''_{-1,-1} & \bar{\delta}''_{-2,-1} & \bar{\delta}''_{-3,-1} & 0 & \bar{\lambda}_m\bar{\beta}''_{2,-1} & \bar{\lambda}_m\bar{\beta}''_{1,-1} \end{pmatrix} \\
&\quad - s_2 \begin{pmatrix} \beta''_{1,-1} \\ \beta''_{2,-1} \\ 0 \\ \delta''_{-3,-1} \\ \delta''_{-2,-1} \\ \delta''_{-1,-1} \end{pmatrix} \begin{pmatrix} \lambda_m\bar{\gamma}''_{-1,2} & \lambda_m\bar{\gamma}''_{-2,2} & \lambda_m\bar{\gamma}''_{-3,2} & 0 & \bar{\alpha}''_{22} & \bar{\alpha}''_{12} \end{pmatrix}) \\
&\quad \cdot (g_2)^{-1}.
\end{aligned}$$

Assume that $\theta \in U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Then $s_2\eta_{*2}\bar{\lambda}_m\bar{\beta}''_{1,-1} - s_2\eta_{*, -1}\bar{\alpha}''_{12} \equiv 0(\text{mod } I_m)$. It follows that $\alpha''_{12} \in I_m$. Hence $-s_2\eta_{*, -1}\bar{\alpha}''_{22} + s_2e_{-1} \equiv 0(\text{mod } I_m)$. By multiplying $\eta'_{-1,*}$ from the left we get that $-s_2\bar{\alpha}''_{22} + s_2\bar{\alpha}''_{11} \in I_m$ which implies $\alpha''_{22} \equiv 1(\text{mod } I_m)$ since $\alpha''_{11} \equiv 1(\text{mod } I_m)$. Let $\hat{\eta}$ be the image of η in $U_{2n}(R_m/I_m, \Lambda_m/(\Lambda_m \cap I_m))$. By Lemma 3.9, $\hat{\eta}_{-2,*} = f_{-2}$ since $\hat{\eta}_{*2} = e_2$. Hence $\gamma''_{-2,3}, \delta''_{-2,-2} - 1, \delta''_{-2,-1} \in I_m$.

Hence $\hat{\eta}$ has the form

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \hat{\alpha}_{13}'' & 0 & 0 & 0 \\ 0 & 1 & \hat{\alpha}_{23}'' & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \hat{\gamma}_{-3,3}'' & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

Clearly

$$\begin{aligned} & \overline{\hat{\alpha}_{13}''} \\ &= \mathbb{h}(\hat{\eta}_{*3}, \hat{\eta}_{*, -1}) \\ &= \mathbb{h}(\hat{\eta}e_3, \hat{\eta}e_{-1}) \\ &= \mathbb{h}(e_3, e_{-1}) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} & \overline{\hat{\alpha}_{23}''} \\ &= \mathbb{h}(\hat{\eta}_{*3}, \hat{\eta}_{*, -2}) \\ &= \mathbb{h}(\hat{\eta}e_3, \hat{\eta}e_{-2}) \\ &= \mathbb{h}(e_3, e_{-2}) \\ &= 0. \end{aligned}$$

Hence $\hat{\alpha}_{13}'' = 0 = \hat{\alpha}_{23}''$ and therefore $\alpha_{13}'', \alpha_{23}'' \in I_m$. Since that is a contradiction, $\theta \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Clearly $\theta_{3*} = f_3$. Set $\epsilon_3 := P_{13}$. Then the first row of ${}^{\epsilon_3}\theta$ equals f_1 . Since ${}^{\epsilon_3}\theta \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, one can proceed as in Part I, case 1.

case 3.5 Assume that $\gamma_{-3,1}'', \gamma_{-3,2}'', \gamma_{-2,1}'', \gamma_{-2,2}'', \gamma_{-1,1}'', \gamma_{-1,2}'', \beta_{1,-2}'', \beta_{1,-1}'', \beta_{2,-2}'', \beta_{2,-1}'', \delta_{-3,-2}'', \delta_{-3,-1}'', \alpha_{11}'' - 1, \alpha_{21}'', \gamma_{-1,3}'', \delta_{-1,-1}'', \delta_{-1,-2}'', \alpha_{12}'', \alpha_{22}'' - 1, \gamma_{-2,3}'', \delta_{-2,-1}'', \delta_{-2,-2}'' - 1, \alpha_{13}'', \alpha_{23}'' \in I_m$.

Since $\eta \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, $\gamma_{-3,3}'' \notin I_m$. By [2], chapter IV, Lemma 3.12, Part II, case 4 there are $x_1, x_2 \in I_m$ such that $\gamma_{-1,3}'' + x_2(x_1\gamma_{-1,3}'' + \gamma_{-2,3}'') \in \text{rad}(R_m) \cap I_m$. Set $\xi_3 := T_{-1,-2}(x_2)T_{-2,-1}(x_1) \in EU_6((R_m, \Lambda_m), (I_m, \Gamma_m))$ and $\theta := \xi_3\eta$. Then $\theta \equiv \eta \pmod{I_m}$, $\theta_{33} = \alpha_{33}'' \equiv 1 \pmod{\text{rad}(R_m) \cap I_m}$ and $\theta_{-1,3} \in \text{rad}(R_m) \cap I_m$. Set $\epsilon_{21} := T_{3,-1}(-1) \in EU_6(R_m, \Lambda_m)$ and $\tau := {}^{\epsilon_{21}}\theta$. Then $\tau_{33} \equiv 1 \pmod{\text{rad}(R_m) \cap I_m}$ and $\tau_{-3,-1} = \theta_{-3,-1} - \theta_{-3,3} \equiv \theta_{-3,3} \equiv \gamma_{-3,3}'' \pmod{I_m}$. Since $\gamma_{-3,3}'' \notin I_m$, it follows that $\tau_{-3,-1} \notin I_m$. Set $\xi_4 := T_{32}(-(\tau_{33})^{-1}\tau_{32})T_{31}(-(\tau_{33})^{-1}\tau_{31})T_{3,-1}(-(\tau_{33})^{-1}\tau_{3,-1})T_{3,-2}(-(\tau_{33})^{-1}\tau_{3,-2}) \in EU_6((R_m, \Lambda_m), (I_m, \Gamma_m))$ and $\chi := \tau\xi_4$. Then χ has the form

$$\left(\begin{array}{ccc|ccc} * & & & * & & \\ 0 & 0 & \alpha_{33}''' & \beta_{3,-3}''' & 0 & 0 \\ \hline & \gamma''' & & & \delta''' & \end{array} \right) = \begin{pmatrix} \alpha''' & \beta''' \\ \gamma''' & \delta''' \end{pmatrix}$$

where $\alpha''', \beta''', \gamma''', \delta''' \in M_3(R_m)$. Further $\alpha'''_{33} \equiv 1 \pmod{\text{rad}(R_m) \cap I_m}$, $\beta'''_{3,-3} \in I_m$ and $\delta'''_{-3,-1} \notin I_m$. One can proceed now as in case 3.1 or case 3.2.

case 4 Assume that $\gamma_{-3,1}, \gamma_{-3,2}, \gamma_{-2,1}, \gamma_{-2,2}, \gamma_{-1,1}, \gamma_{-1,2}, \beta_{3,-3}, \beta_{3,-2} \in I_m$ and $\beta_{3,-1} \notin I_m$.

Set $\epsilon_{11} := T_{21}(1) \in EU_6(R_m, \Lambda_m)$ and $\rho := {}^{\epsilon_{11}}\sigma$. Clearly $\rho_{-3,1}, \rho_{-3,2}, \rho_{-2,1}, \rho_{-2,2}, \rho_{-1,1}, \rho_{-1,2} \in I_m$. Further

$$\rho_{3*} = \begin{pmatrix} 0 & 0 & 1 & \beta_{3,-3} & \beta_{3,-2} + \beta_{3,-1} & \beta_{3,-1} \end{pmatrix}.$$

Since $\beta_{3,-2} \in I_m$ and $\beta_{3,-1} \notin I_m$, $\beta_{3,-2} + \beta_{3,-1} \notin I_m$. One can proceed now as in case 3.

case 5 Assume that $\gamma_{-3,1}, \gamma_{-3,2}, \gamma_{-2,1}, \gamma_{-2,2}, \gamma_{-1,1}, \gamma_{-1,2}, \beta_{3,-3}, \beta_{3,-2}, \beta_{3,-1} \in I_m$. One can proceed as in case 3 (σ has the same properties as ζ in case 3).

Part III Assume that $h \in CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$.

This part corresponds to Proposition 3.3 in [2], chapter IV. By [2], chapter IV, Corollary 3.4, there is an elementary short or long root element $T_{ij}(x) \in EU_{2n}(R_m, \Lambda_m)$ such that $[h, T_{ij}(x)] \notin U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$ since $h \notin CU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Since $h \in CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, $[h, T_{ij}(x)] \in U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. It follows from Lemma 4.6 that there is an elementary matrix $g_0 \in U_0$ such that $[h, g_0] \in U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m)) \setminus U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$ (note that $J(h) \subseteq I_m$ and $h_{kk} \equiv h_{ll} \pmod{I_m} \forall k, l$ since $h \in CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$). Set $\sigma := [h, g_0]$. By Lemma 4.7, there is an $\epsilon_1 \in EU_{2n}(R_m, \Lambda_m)$ such that $y_1 := ({}^{\epsilon_1}\sigma)_{11}$ is invertible. Clearly ${}^{\epsilon_1}\sigma \in U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m)) \setminus U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Set $\omega := {}^{\epsilon_1}\sigma$, $\xi_1 := T_{-2,1}(-\omega_{-2,1}(y_1)^{-1}) \dots T_{21}(-\omega_{21}(y_1)^{-1}) \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$ and $\xi_2 := T_{12}(-(y_1)^{-1}\omega_{12}) \dots T_{1,-2}(-(y_1)^{-1}\omega_{1,-2}) \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Then $\tau := \xi_1\omega\xi_2$ has the form

$$\begin{pmatrix} y_1 & 0 & y_2 \\ 0 & A & 0 \\ y_3 & 0 & y_4 \end{pmatrix}$$

where $y_2, y_3, y_4 \in R_m$ and $A \in M_{2n-2}(R_m)$.

case 1 Assume that $y_3(y_1)^{-1} \in \Gamma_m$ and $(y_1)^{-1}y_2 \in \bar{\lambda}_m\Gamma_m$.

Set $\xi_3 := T_{-1,1}(-y_3(y_1)^{-1}) \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$ and $\xi_4 := T_{1,-1}(-(y_1)^{-1}y_2) \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Then $\zeta := \xi_3\tau\xi_4$ has the form

$$\begin{pmatrix} y_1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & y_5 \end{pmatrix}$$

where $y_5 \in R_m$. Clearly $\zeta \in U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ but $\zeta \notin U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Hence there is an $l \in \{2, \dots, -2\}$ such that $|\zeta_{*l}| \notin \Gamma_m$.

case 1.1 Assume that $\epsilon(l) = 1$.

There are a $b' \in R$ and a $t' \in S_m$ such that $y_1|\zeta_{*l}|\bar{y}_1 - \zeta_{-l,l}\bar{y}_1 + \lambda_m \overline{\zeta_{-l,l}\bar{y}_1} = \frac{b'}{t'}$. Set $t := \frac{t'}{1} \in R_m$ and $g_1 := T_{l,-1}(s_1 s_2 t) \in U_1$. One can show that $[\zeta, g_1]$ equals

$$\begin{aligned} & T_{2,-1}(s_1 s_2 t \zeta_{2l} \bar{y}_1) \cdot \\ & \vdots \\ & T_{l-1,-1}(s_1 s_2 t \zeta_{(l-1)l} \bar{y}_1) \cdot \\ & T_{l,-1}(s_1 s_2 t \zeta_{ll} \bar{y}_1 - s_1 s_2 t) \cdot \\ & T_{l+1,-1}(s_1 s_2 t \zeta_{(l+1)l} \bar{y}_1) \cdot \\ & \vdots \\ & T_{-2,-1}(s_1 s_2 t \zeta_{-2,l} \bar{y}_1) \cdot \\ & T_{1,-1}(z) \end{aligned}$$

where $z = \bar{\lambda}_m \overline{s_1 s_2 t}(y_1|\zeta_{*l}|\bar{y}_1 - \zeta_{-l,l}\bar{y}_1 + \lambda_m \overline{\zeta_{-l,l}\bar{y}_1})s_1 s_2 t$. Since $|\zeta_{*l}| \notin \Gamma_m$ and y_1 is invertible, $y_1|\zeta_{*l}|\bar{y}_1 \notin \Gamma_m$. Since $\zeta_{-l,l} \in I_m$, $-\zeta_{-l,l}\bar{y}_1 + \lambda_m \overline{\zeta_{-l,l}\bar{y}_1} \in (\Gamma_m)_{\min} \subseteq \Gamma_m$. Since $s_1 s_2 t$ is invertible, it follows that $\overline{s_1 s_2 t}(y_1|\zeta_{*l}|\bar{y}_1 - \zeta_{-l,l}\bar{y}_1 + \lambda_m \overline{\zeta_{-l,l}\bar{y}_1})s_1 s_2 t \notin \Gamma_m$. Hence $z \notin \bar{\lambda}_m \Gamma_m$. Set

$$\begin{aligned} & \xi_5 := \\ & T_{-2,-1}(-s_1 s_2 t \zeta_{-2,l} \bar{y}_1) \cdot \\ & \vdots \\ & T_{l+1,-1}(-s_1 s_2 t \zeta_{(l+1)l} \bar{y}_1) \cdot \\ & T_{l,-1}(-(s_1 s_2 t \zeta_{ll} \bar{y}_1 - s_1 s_2 t)) \cdot \\ & T_{l-1,-1}(-s_1 s_2 t \zeta_{(l-1)l} \bar{y}_1) \cdot \\ & \vdots \\ & T_{2,-1}(-s_1 s_2 t \zeta_{2l} \bar{y}_1) \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m)) \end{aligned}$$

and $g_2 := T_{1,-1}(z) \in U_2$. Then

$$(\xi_5[\xi_3 \xi_1(\epsilon_1[h, g_0])\xi_2 \xi_4, g_1]) = g_2.$$

Note that $g_2 \notin U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$ since $z \notin \bar{\lambda}_m \Gamma_m$. Set $g'_i := \psi_m(g_i) \ \forall i \in \{0, 1, 2\}$ and $\epsilon'_1 := \psi_m(\epsilon_1)$. Then

$$[\epsilon'_1[h', g'_0], g'_1] = g'_2.$$

case 1.2 Assume that $\epsilon(l) = -1$.

This case can be treated similarly.

case 2 Assume that $y_3(y_1)^{-1} \notin \Gamma_m$.

case 2.1 Assume that $|\tau_{*l}| \in \Gamma_m \ \forall l \in \{2, \dots, -2\}$.

Set $\chi := T_{-1,1}(-y_3(y_1)^{-1}) \in EU_{2n}(R_m, \Lambda_m)$ (one checks easily that $-y_3(y_1)^{-1} \in \Lambda_m$). Then $\zeta := \chi\tau$ has the form

$$\begin{pmatrix} y_1 & 0 & y_2 \\ 0 & A & 0 \\ 0 & 0 & y_5 \end{pmatrix}$$

where $y_5 \in R_m$. There is an $b' \in R$ and a $t' \in S_m$ such that $y_3(y_1)^{-1} = \frac{b'}{t'}$. Set $t := \frac{t'}{1} \in R_m$ and $g_1 := T_{12}(s_1 s_2 t) \in U_1$. Using the equality $[\alpha\beta, \gamma] = {}^\alpha[\beta, \gamma][\alpha, \gamma]$ one gets that $[\tau, g_1] = [\chi^{-1}\zeta, g_1] = \chi^{-1}[\zeta, g_1][\chi^{-1}, g_1]$. It is easy to show that $[\zeta, g_1] \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$ and hence $\chi^{-1}[\zeta, g_1] \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. On the other hand $[\chi^{-1}, g_1] = T_{-1,2}(y_3(y_1)^{-1} s_1 s_2 t) T_{-2,2}(-\overline{s_1 s_2 t} y_3(y_1)^{-1} s_1 s_2 t)$, by (R6) in Lemma 3.12. Set $\xi_3 := T_{-1,2}(-y_3(y_1)^{-1} s_1 s_2 t) (\chi^{-1}[\zeta, g_1])^{-1} \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$ and $g_2 := T_{-2,2}(-\overline{s_1 s_2 t} y_3(y_1)^{-1} s_1 s_2 t) \in U_2$. Since $y_3(y_1)^{-1} \notin \Gamma_m$, $g_2 \notin U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Clearly

$$(\xi_3[\xi_1({}^{\epsilon_1}[h, g_0])\xi_2, g_1]) = g_2.$$

As above, push this equation into $U_{2n}(R_m, \Lambda_m)/U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$ by applying ψ_m .

case 2.2 Assume that there is an $l \in \{2, \dots, -2\}$ such that $|\tau_{*l}| \notin \Gamma_m$.

Choose a $p \in \{2, \dots, -2\}$ such that $p \neq \pm l$ and set $g_1 := T_{lp}(s_1) \in U_1$. Then $\zeta := [\tau, g_1]$ has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $B \in M_{2n-2}(R_m)$. Since $\tau \in U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, $\zeta \in U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. By Lemma 4.6, $|\zeta_{*p}| \notin \Gamma_m$. Hence $\zeta \notin U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$ and thus one can proceed as in case 1.

case 3 Assume that $(y_1)^{-1} y_2 \notin \bar{\lambda}_m \Gamma_m$.

See case 2. □

Lemma 4.9 *Let (I, Ω) be a form ideal of (R, Λ) and m a maximal ideal of C such that $I \cap C \subseteq m$. Then the following is true.*

(4.9.1) *If $U' \in A'$ and $g' \in \phi_m(\psi(EU_{2n}(R, \Lambda)))$ is the nontrivial image of an elementary matrix in $EU_{2n}(R, \Lambda)$, then*

$$V' \subseteq {}^{U'}g'$$

for some $V' \in B'$.

(4.9.2) If $V' \in B'$ and $d' \in \psi_m(EU_{2n}(R_m, \Lambda_m))$ is the image of an elementary matrix in $EU_{2n}(R_m, \Lambda_m)$, then

$$g' \in {}^{d'}V'$$

for some nontrivial image $g' \in \phi_m(\psi(EU_{2n}(R, \Lambda)))$ of an elementary matrix in $EU_{2n}(R, \Lambda)$.

Proof Follows from the relations (R1)-(R6) in Lemma 3.12. \square

Corollary 4.10 Let (I, Ω) be a form ideal of (R, Λ) and m a maximal ideal of C such that $I \cap C \subseteq m$. If $U' \in A'$, $d' \in \psi_m(EU_{2n}(R_m, \Lambda_m))$ and $g' \in \phi_m(\psi(EU_{2n}(R, \Lambda)))$ is the nontrivial image of an elementary matrix in $EU_{2n}(R, \Lambda)$, then

$$V' \subseteq {}^{U'}({}^{d'}g')$$

for some $V' \in B'$.

Proof If $d' = 1$, then we are done, by (4.9.1). Assume $d' \neq 1$ and write d' as a product $d'_k \dots d'_1$ of nontrivial images of elementary matrices in $EU_{2n}(R_m, \Lambda_m)$. We proceed by induction on k .

case 1 Assume that $k = 1$. Since (A', B') is a supplemented base for $\psi_m(EU_{2n}(R_m, \Lambda_m))$, there is a $U'_1 \in A'$ such that ${}^{d'_1}U'_1 \subseteq U'$. Clearly

$$\begin{aligned} & {}^{U'}({}^{d'_1}g') \\ \supseteq & {}^{U'}({}^{d'_1}({}^{U'_1}g')) \\ \stackrel{(4.9.1)}{\supseteq} & {}^{U'}({}^{d'_1}V') \text{ (for some } V' \in B') \\ \stackrel{(4.9.2)}{\supseteq} & {}^{U'}g'' \text{ (for some nontrivial image } g'' \text{ of an elementary matrix in } EU_{2n}(R, \Lambda)) \\ \stackrel{(4.9.1)}{\supseteq} & V'_1 \text{ (for some } V'_1 \in B'). \end{aligned}$$

case 2 Assume that $k > 1$. Set $h' := d'_{k-1} \dots d'_1$. Thus $d' = d'_k \dots d'_1 = d'_k h'$. We can assume by induction on k that given $U'_1 \in A'$, ${}^{U'_1}({}^{h'}g') \supseteq V'$ for some $V' \in B'$. Now we proceed similarly to case 1, replacing g' by ${}^{h'}g'$ and d'_1 by d'_k . Here are the details. Choose $U'_1 \in A'$ such that ${}^{d'_k}\phi(U'_1) \subseteq U'$. Clearly

$$\begin{aligned} & {}^{U'}({}^{d'_k h'}g') \\ \supseteq & {}^{U'}({}^{d'_k}({}^{U'_1}({}^{h'}g')))) \\ \stackrel{I.A.}{\supseteq} & {}^{U'}({}^{d'_k}V') \\ \stackrel{(4.9.2)}{\supseteq} & {}^{U'}g'' \text{ (for some nontrivial image } g'' \text{ of an elementary matrix in } EU_{2n}(R, \Lambda)) \\ \stackrel{(4.9.1)}{\supseteq} & V'_1 \text{ (for some } V'_1 \in B'). \end{aligned}$$

\square

Theorem 4.11 Let n and (R, Λ) be as in the third paragraph of this section. Let H be a subgroup of $U_{2n}(R, \Lambda)$. Then

H is normalized by $EU_{2n}(R, \Lambda) \Leftrightarrow$

$\exists!$ form ideal (I, Γ) such that $EU_{2n}((R, \Lambda), (I, \Gamma)) \subseteq H \subseteq CU_{2n}((R, \Lambda), (I, \Gamma))$.

Proof

\Rightarrow :

Assume that H is normalized by $EU_{2n}(R, \Lambda)$. We have to show existence and uniqueness of a form ideal (I, Γ) such that

$$EU_{2n}((R, \Lambda), (I, \Gamma)) \subseteq H \subseteq CU_{2n}((R, \Lambda), (I, \Gamma)).$$

existence

Set $I := \{x \in R \mid T_{12}(x) \in H\}$ and $\Gamma := \{y \in \Lambda \mid T_{-1,1}(y) \in H\}$. Then (I, Γ) is a form ideal, $EU_{2n}((R, \Lambda), (I, \Gamma)) \subseteq H$ and (I, Γ) is maximal with this property (i.e. if $EU_{2n}((R, \Lambda), (I', \Gamma')) \subseteq H$, then $I' \subseteq I$ and $\Gamma' \subseteq \Gamma$). It remains to show that $H \subseteq CU_{2n}((R, \Lambda), (I, \Gamma))$. The proof is by contradiction. Suppose $H \not\subseteq CU_{2n}((R, \Lambda), (I, \Gamma))$. Then the image \hat{H} of H in $U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$ contains a noncentral element \hat{h} by the definition of $CU_{2n}((R, \Lambda), (I, \Gamma))$. By Lemma 4.1 there is a maximal ideal m of C such that $I \cap C \subseteq m$ and $h' := \phi_m(\hat{h})$ is noncentral in $U_{2n}(R_m, \Lambda_m)/U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Choose an $s_0 \in S_m$ with the properties (1) and (2) in Lemma 4.2, let (A, B) be the supplemented base for $EU_{2n}(R, \Lambda)$ defined in Lemma 4.4 and set $(A', B') := \phi_m(\psi(A, B))$. Choose an $U' \in A'$. By Lemma 4.8 there is a $k \in \mathbb{N}$ and elements $g'_0, \dots, g'_k \in \phi_m(\psi(EU_{2n}(R, \Lambda)))$, $\epsilon'_0, \dots, \epsilon'_k \in \psi_m(EU_{2n}(R_m, \Lambda_m))$ and $l_1, \dots, l_k \in \{-1, 1\}$ such that g'_k is the nontrivial image of an elementary matrix in $EU_{2n}(R, \Lambda)$,

$$\epsilon'_k([\epsilon'_{k-1}(\dots \epsilon'_2([\epsilon'_1([\epsilon'_0 h', g'_0]^{l_1}), g'_1]^{l_2}) \dots), g'_{k-1}]^{l_k}) = g'_k \quad (4.11.1)$$

and

$$d'_i g'_i \in U' \quad \forall i \in \{0, \dots, k\}$$

where $d'_i = (\epsilon'_i \dots \epsilon'_0)^{-1} \forall i \in \{0, \dots, k\}$. By conjugating (4.11.1) by $d'_k = (\epsilon'_k \dots \epsilon'_0)^{-1}$ we get

$$[\dots [[h', d'_0 g'_0]^{l_1}, d'_1 g'_1]^{l_2} \dots, d'_{k-1} g'_{k-1}]^{l_k} = d'_k g'_k. \quad (4.11.2)$$

By Corollary 4.10 there is a $V' \in B'$ such that

$$V' \subseteq U' (d'_k g'_k). \quad (4.11.3)$$

Let $\hat{U} \in \hat{A} := \psi(A)$ and $\hat{V} \in \hat{B} := \psi(B)$ such that $\phi_m(\hat{U}) = U'$ and $\phi_m(\hat{V}) = V'$. Clearly we may assume that $\hat{V} \subseteq \hat{U}$ since $\hat{V} \cap \hat{U}$ contains a member of \hat{B} . Since $d'_i g'_i \in U' = \phi_m(\hat{U}) \quad \forall i \in \{0, \dots, k\}$, there are $x_0, \dots, x_k \in \hat{U}$ such that $\phi_m(x_i) = d'_i g'_i \quad \forall i \in \{0, \dots, k\}$. Set

$$x := [\dots [[\hat{h}, x_0]^{l_1}, x_1]^{l_2} \dots, x_{k-1}]^{l_k}.$$

Clearly the l.h.s. of (4.11.2) equals $\phi_m(x)$. Further $x \in \hat{H}$ since \hat{H} is normalized by $\psi(EU_{2n}(R, \Lambda))$ and $x \in M := \psi(U_{2n}((R, \Lambda), (s_0 R, s_0 \Lambda)))$ since $x_{k-1} \in \hat{U} \subseteq M$ and M is normal. It follows that $\hat{U}x \subseteq \hat{H} \cap M$. By (4.11.2) and (4.11.3) we have

$$\phi_m(\hat{V}) \subseteq \phi_m(\hat{U})\phi_m(x) = \phi_m(\hat{U}x) \subseteq \phi_m(\hat{H} \cap M). \quad (4.11.4)$$

By Lemma 4.2, ϕ_m is injective on M . Hence it follows from (4.11.4) that $\hat{V} \subseteq \hat{H}$ (note that $\hat{V} \subseteq \hat{U} \subseteq M$). Let $V = EU_{2n}(Rx s_0 R, \Gamma(x s_0)) \in B$, where $x \in R, x s_0 \notin I$

or $x \in \Lambda$, $xs_0 \notin \Gamma$, such that $\hat{V} = \psi(V)$. It follows that $V \subseteq H \cdot U_{2n}((R, \Lambda), (I, \Gamma))$. This implies $EU_{2n}((R, \Lambda), (Rxs_0R, \Gamma(xs_0))) \subseteq H \cdot U_{2n}((R, \Lambda), (I, \Gamma))$, since both H and $U_{2n}((R, \Lambda), (I, \Gamma))$ are normalized by $EU_{2n}(R, \Lambda)$. Hence

$$\begin{aligned}
& EU_{2n}((R, \Lambda), (Rxs_0R, \Gamma(xs_0))) \\
&= [EU_{2n}(R, \Lambda), EU_{2n}((R, \Lambda), (Rxs_0R, \Gamma(xs_0)))] \\
&\subseteq [EU_{2n}(R, \Lambda), H \cdot U_{2n}((R, \Lambda), (I, \Gamma))] \\
&\subseteq [EU_{2n}(R, \Lambda), H](^H[EU_{2n}(R, \Lambda), U_{2n}((R, \Lambda), (I, \Gamma))]) \\
&= [EU_{2n}(R, \Lambda), H](^HEU_{2n}((R, \Lambda), (I, \Gamma))) \\
&\subseteq H
\end{aligned}$$

by Lemma 3.16. But this contradicts the maximality of (I, Γ) since clearly $Rxs_0R \not\subseteq I$ or $\Gamma(xs_0) \not\subseteq \Gamma$. Thus $H \subseteq CU_{2n}((R, \Lambda), (I, \Gamma))$.

uniqueness

Assume that

$$EU_{2n}((R, \Lambda), (I, \Gamma)) \subseteq H \subseteq CU_{2n}((R, \Lambda), (I, \Gamma))$$

and

$$EU_{2n}((R, \Lambda), (I', \Gamma')) \subseteq H \subseteq CU_{2n}((R, \Lambda), (I', \Gamma')).$$

It follows that

$$\begin{aligned}
& EU_{2n}((R, \Lambda), (I, \Gamma)) \\
&= [EU_{2n}(R, \Lambda), EU_{2n}((R, \Lambda), (I, \Gamma))] \\
&\subseteq [EU_{2n}(R, \Lambda), CU_{2n}((R, \Lambda), (I', \Gamma'))] \\
&= EU_{2n}((R, \Lambda), (I', \Gamma')).
\end{aligned}$$

It is easy to deduce that $I \subseteq I'$ and $\Gamma \subseteq \Gamma'$. By symmetry it follows that $I = I'$ and $\Gamma = \Gamma'$.

\Leftarrow :

Suppose that

$$EU_{2n}((R, \Lambda), (I, \Gamma)) \subseteq H \subseteq CU_{2n}((R, \Lambda), (I, \Gamma)).$$

Then

$$[EU_{2n}(R, \Lambda), H] \subseteq [EU_{2n}(R, \Lambda), CU_{2n}((R, \Lambda), (I, \Gamma))] = EU_{2n}((R, \Lambda), (I, \Gamma)) \subseteq H$$

and hence H is normalized by $EU_{2n}(R, \Lambda)$. \square

Definition 4.12 Let R be a ring, $r \mapsto \bar{r}$ an involution on R and $\lambda \in \text{center}(R)$. Then we call R *quasi-finite* if it is a direct limit of subrings R_i ($i \in \Phi$ where Φ is some index set) which are almost commutative, involution invariant and contain λ (recall that a ring is called almost commutative if it is module finite over its center).

Lemma 4.13 (A. Bak) *Let T be an almost commutative ring, $t \mapsto \bar{t}$ an involution on T and $\lambda \in \text{center}(T)$. Then T is a direct limit of involution invariant subrings T_j ($j \in \Psi$) containing λ such that for any $j \in \Psi$, T_j is a Noetherian C_j -module where C_j is the subring of T_j consisting of all finite sums of elements of the form $c\bar{c}$ and $-c\bar{c}$ where c ranges over some subring $C'_j \subseteq \text{center}(T_j)$.*

Proof Denote the center of T by C . Since T is almost commutative, there is an $q \in \mathbb{N}$ and elements $x_1, \dots, x_q \in T$ such that $T = Cx_1 + \dots + Cx_q$. For each $k, l \in \{1, \dots, q\}$ there are $a_1^{(kl)}, \dots, a_q^{(kl)} \in C$ such that $x_k x_l = \sum_{p=1}^q a_p^{(kl)} x_p$. Further for each $k \in \{1, \dots, q\}$ there are $b_1^{(k)}, \dots, b_q^{(k)} \in C$ such that $\bar{x}_k = \sum_{p=1}^q b_p^{(k)} x_p$. Finally there are $c_1, \dots, c_q \in C$ such that $\lambda = \sum_{p=1}^q c_p x_p$. Set

$$K := \mathbb{Z}[a_p^{(kl)}, \overline{a_p^{(kl)}}, b_p^{(k)}, \overline{b_p^{(k)}}, c_p, \overline{c_p} | k, l, p \in \{1, \dots, q\}].$$

One checks easily that C is a K -algebra and the direct limit of all involution invariant K -subalgebras A_j ($j \in \Psi$) of C which are finitely generated over K . For any $j \in \Psi$ set $T_j := A_j + A_j x_1 + \dots + A_j x_q$. One checks easily that each T_j is an involution invariant subring of T containing λ . Further $\varinjlim_j T_j = T$. Fix a $j \in \Psi$ and let C_j

denote the subring of A_j consisting of all finite sums of elements of the form $a\bar{a}$ and $-a\bar{a}$ where $a \in A_j$. We will show that T_j is a Noetherian C_j -module. Clearly A_j is a finitely generated \mathbb{Z} -algebra and hence also a finitely generated C_j -algebra. Since for any $a \in A_j$

$$a + \bar{a} = (a + 1)(\bar{a} + 1) - a\bar{a} - 1,$$

C_j contains all sums $a + \bar{a}$ where $a \in A_j$. Since any $a \in A_j$ is root of the monic polynomial $X^2 - (a + \bar{a})X + a\bar{a}$, A_j is an integral extension of C_j . Since A_j is an integral extension of C_j and a finitely generated C_j -algebra, A_j is a finitely generated module over C_j by [6], chapter VII, Proposition 1.2. Since T_j is finitely generated over A_j , it is a finitely generated C_j -module. Since K is a Noetherian ring, A_j is a Noetherian ring (by Hilbert's Basis Theorem) and hence C_j is a Noetherian ring (by the Eakin-Nagata Theorem). Thus T_j is a Noetherian C_j -module. \square

Theorem 4.14 *Let $n \geq 3$ and (R, Λ) a form ring where R is quasi-finite. Let H be a subgroup of $U_{2n}(R, \Lambda)$. Then*

H is normalized by $EU_{2n}(R, \Lambda) \Leftrightarrow$

$\exists!$ form ideal (I, Γ) such that $EU_{2n}((R, \Lambda), (I, \Gamma)) \subseteq H \subseteq CU_{2n}((R, \Lambda), (I, \Gamma))$.

Proof Let (I, Γ) denote the level of H , i.e. the largest form ideal such that $EU_{2n}((R, \Lambda), (I, \Gamma)) \subseteq H$. We will show that $H \subseteq CU_{2n}((R, \Lambda), (I, \Gamma))$, i.e. $[\sigma, \tau] \in U_{2n}((R, \Lambda), (I, \Gamma))$ for any $\sigma \in H$ and $\tau \in U_{2n}(R, \Lambda)$. Let $\sigma \in H$ and $\tau \in U_{2n}(R, \Lambda)$. Since R is quasi-finite, it is the direct limit of almost commutative, involution invariant subrings R_i ($i \in \Phi$) containing λ . By the previous lemma each R_i is the direct limit of involution invariant subrings R_{ij} ($j \in \Psi_i$) containing λ such that for

any $j \in \Psi_i$, R_{ij} is a Noetherian C_{ij} -module where C_{ij} is the subring of R_{ij} consisting of all finite sums of elements of the form $c\bar{c}$ and $-c\bar{c}$ where c ranges over some subring $C'_{ij} \subseteq \text{center}(R_{ij})$. If $i \in \Phi$ and $j \in \Psi_i$, set $\Lambda_{ij} := \Lambda \cap R_{ij}$. One checks easily that (R_{ij}, Λ_{ij}) is a form ring. Clearly there is an $i \in \Phi$ and a $j \in \Psi_i$ such that $\sigma, \tau \in U_{2n}(R_{ij}, \Lambda_{ij})$. Set $H_{ij} := U_{2n}(R_{ij}, \Lambda_{ij}) \cap H$. Then $\sigma \in H_{ij}$ and H_{ij} is normalized by $EU_{2n}(R_{ij}, \Lambda_{ij})$. Let (I_{ij}, Γ_{ij}) denote the level of H_{ij} . Then obviously $I_{ij} \subseteq I$ and $\Gamma_{ij} \subseteq \Gamma$. By Theorem 4.11,

$$H_{ij} \subseteq CU_{2n}((R_{ij}, \Lambda_{ij}), (I_{ij}, \Gamma_{ij})).$$

Hence $[\sigma, \tau] \in U_{2n}((R_{ij}, \Lambda_{ij}), (I_{ij}, \Gamma_{ij})) \subseteq U_{2n}((R, \Lambda), (I, \Gamma))$. Thus we have shown that

$$EU_{2n}((R, \Lambda), (I, \Gamma)) \subseteq H \subseteq CU_{2n}((R, \Lambda), (I, \Gamma))$$

where (I, Γ) is the level of H . The uniqueness of (I, Γ) and the implication \Leftarrow follow from the standard commutator formulas (see the proof of Theorem 4.11). \square

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